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GENERALIZED CASSON INVARIANTS FOR SO(3), U(2), Spin(4), AND SO(4)

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ABSTRACT. We investigate Casson-type invariants corresponding to the low-rank groups SO(3), SU(2) $\times S^1$, U(2), Spin(4) and SO(4). The invariants are defined following an approach similar to those of K. Walker and S. Cappell, R. Lee, and E. Miller. We obtain a description for each of the invariants in terms of the SU(2)-invariant. Thus, all of them may be calculated using formulae for the SU(2)-invariant. In defining these invariants, we offer methods which should prove useful for studying the invariants for other non-simply-connected groups once the invariants for the simply-connected covering groups are known.

0.1. INTRODUCTION

In a series of lectures [C] in 1985, Casson defined an invariant of integral homology 3-spheres which was a refinement of the μ-invariant of Rochlin. Roughly, the invariant is the number of "signed" equivalence classes of representations of the fundamental group in SU(2). The idea is to take a Heegaard splitting of the manifold and define an intersection number for the associated representation spaces. The difficulty in doing this arises from the fact that the representation spaces involved are not manifolds; rather they are varieties with singular points corresponding to the reducible representations of the fundamental group. Casson overcomes this difficulty by observing that for integral homology spheres the only reducible representation of the fundamental group in the trivial one, and the representation spaces for the two handlebodies of the Heegaard decomposition are already transverse at the trivial representation.

After Casson announced his invariant, several mathematicians sought to extend this idea to define invariants for other manifolds and other Lie groups. In [A], Atiyah suggested that this process could be carried out for SU(n) and by implication for any compact Lie group. In [BN], Boyer and Nicas defined an invariant for a larger class of manifolds including rational homology spheres. They explained that for manifolds which are not sufficiently large one may simply ignore the irreducible representations, as Casson did. In [BL], Boyer and Lines defined another generalization of Casson's invariant counting SU(2)-representations of integral homology lens spaces. This invariant included a term measuring the effect of the reducible representations of the fundamental group.

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Finally, in his doctoral dissertation at Berkeley [W], Walker extended Casson's definition to give an invariant of rational homology 3-spheres which fully incorporated the reducible representations. Walker used work by Goldman [G1 and G2] on the natural symplectic structures of the representation spaces involved to count the number of "extra" intersection points arising in the normal bundle of the space of reducible representations inside the space of SU(2)-representations under isotopy. This enabled him to accurately count the irreducible representations without ignoring the reducible ones. He obtains an invariant which agrees with those of Casson and of Boyer and Lines for appropriate manifolds but is more sophisticated than that of Boyer and Nicas.

Both Casson and Walker give Dehn surgery formulae for calculating the invariants. These formulae allow us to compute the invariant combinatorially given a description of the manifold as the result of a sequence of Dehn surgeries on a family of knots in S^3.

A definition for the $\mathrm{SU}(n)$-invariant following the approach of Walker was offered by Cappell, Lee and Miller in [CLM]. However no formula for calculating the invariants has been announced as of yet.

The purpose of this paper is to investigate the invariants corresponding to the low-rank groups $\mathrm{SO}(3)$, $\mathrm{SU}(2)\times S^1$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$. We define these invariants following an approach similar to those of Walker and Cappell, Lee, and Miller. We obtain a description for each of the invariants in terms of the SU(2)-invariant. (See Theorems 3.4, 4.3, 4.6, 4.7, and 4.8.) Thus, all of them may be calculated using the surgery formulae of Casson and Walker.

In defining these invariants, we answer Atiyah's question for low-rank groups. We also offer methods which should prove useful for studying the invariants for other non-simply-connected groups once the invariants for the simply-connected covering groups are known.

The paper is outlined as follows: in §1, we provide background information applicable to all of the groups. We define the representation spaces involved and determine their stratifications. The symplectic geometry which will be used in defining the invariants is outlined, and orientation conventions are established.

The SO(3)-invariant is defined in §2 and calculated in §3. The definition is closely related to that of the SU(2)-invariant, although a bit more care is needed when dealing with representations into O(2) and the group of diagonal matrices. We explain how to lift the entire theory to SU(2). This enables us to derive a surgery formula for the invariant in §3.2. In fact, the SO(3)-invariant obeys the same surgery formula as the SU(2)-invariant of Walker; thus we see that the invariants are identical.

Section 4 is devoted to the $\mathrm{SU}(2)\times S^1$-, U(2)-, Spin(4)-, and SO(4)-invariants. Relating these groups to SU(2) and SO(3), we are able to quickly define the invariants and express them in terms of the SU(2)-invariant. We find that the $\mathrm{SU}(2)\times S^1$- and U(2)-invariants are a scalar multiple of the SU(2)-invariant, while the Spin(4)- and SO(4)-invariants are a scalar multiple of the square of the SU(2)-invariant.

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1. Intersections of representation spaces

1.1. **Casson-type invariants.** Let M be a rational homology sphere ($\mathbb{Q}HS$) with Heegaard decomposition (H_1, H_2, Σ). (Thus, H_1 and H_2 are handlebodies, Σ is a Riemann surface, and we have $M = H_1 \cup H_2$, $\Sigma = \partial H_1 = \partial H_2 = H_1 \cap H_2$.) Let Σ^* be Σ minus a disk. Let G be a compact Lie group. The diagram of fundamental groups

$$\begin{array}{ccccc} & & \pi_1(H_1) & & \\ & \nearrow & & \searrow & \\ \pi_1(\Sigma^*) \to \pi_1(\Sigma) & & & & \pi_1(M) \\ & \searrow & & \nearrow & \\ & & \pi_1(H_2) & & \end{array}$$

induces a diagram

$$\begin{array}{ccccc} & & Q^{\#}_{1,G} & & \\ & \swarrow & & \nwarrow & \\ R^{*\#}_G \leftarrow R^{\#}_G & & & & Q^{\#}_{1,G} \cap Q^{\#}_{2,G} = \mathrm{Hom}(\pi_1(M), G) \\ & \nwarrow & & \swarrow & \\ & & Q^{\#}_{2,G} & & \end{array}$$

of the corresponding spaces of homomorphisms into G. Note that all maps in the first diagram are surjective, while all maps in the second diagram are injective. Note too that $Q^{\#}_{1,G} \cap Q^{\#}_{2,G} = \mathrm{Hom}(\pi_1(M), G)$ is an immediate consequence of the Seifert-VanKampen theorem. Finally, since G acts naturally by conjugation on these spaces, we obtain the diagram

$$\begin{array}{ccccc} & & Q_{1,G} & & \\ & \swarrow & & \nwarrow & \\ R^{*}_G \leftarrow R_G & & & & Q_{1,G} \cap Q_{2,G} \\ & \nwarrow & & \swarrow & \\ & & Q_{2,G} & & \end{array}$$

of equivalence classes under conjugation.

We wish to count the number of signed equivalence classes of irreducible (or nearly irreducible, if G is not simply connected) representations of $\pi_1(M)$ in G. Equivalently, since $\mathrm{Hom}(\pi_1(M), G)/G = Q_{1,G} \cap Q_{2,G}$, we wish to define some sort of intersection number for $Q_{1,G}$ and $Q_{2,G}$ in R_G and show that this number depends only upon M and G.

We develop this theory for the groups $\mathrm{SO}(3)$, $\mathrm{SU}(2) \times S^1$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$.

1.2. **Stratified symplectic spaces.** The difficulty in defining the required intersection numbers comes from the fact that the representation spaces defined in §1 are not manifolds; rather, they are orbifolds. In this section we determine the stratifications of these spaces. We also study their symplectic geometry, which will be the key to obtaining well-defined invariants.

Let π be a finitely presented group, and let G be a compact Lie group. In [G], W. Goldman explains that the sets of the form

$$\mathrm{Hom}(\pi, Z(Z(X)))^- / N_G(Z(Z(X)))$$

give a natural stratification of $\operatorname{Hom}(\pi, G)/G$. Here $Z(X)$ is the centralizer of a subgroup $X \subset G$ in G, $Z(Z(X))$ is the double centralizer, $\operatorname{Hom}(\pi, Z(Z(X)))^-$ is the set of all ρ in the $\operatorname{Hom}(\pi, Z(Z(X)))$ at which the dimension of $Z(\rho(\pi))/Z(Z(X))$ is minimal, and $N_G(Z(Z(X)))$ is the normalizer of $Z(Z(X))$ in G.

We are particularly interested in the stratifications of R_G, and $Q_{i,G}$ where $G = \mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathrm{SU}(2) \times S^1$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$.

We begin with $\mathrm{SU}(2)$. Fix a maximal torus S_0^1 of $\mathrm{SU}(2)$. Up to conjugation, the centralizers $Z(Z(X))$ in $\mathrm{SU}(2)$ are $\mathrm{SU}(2)$, S_0^1, and $\mathbb{Z}/2 = Z(\mathrm{SU}(2))$. Applying Goldman's theorem, we obtain the stratification

$$R_{\mathrm{SU}(2)} \supset S_{\mathrm{SU}(2)} \supset Z_{\mathrm{SU}(2)},$$

where $S_{\mathrm{SU}(2)}$ is the set of conjugacy classes of representations in S_0^1 and $Z_{\mathrm{SU}(2)}$ is the set of representations in the center of $\mathrm{SU}(2)$. By the above, $S_{\mathrm{SU}(2)} = (S_0^1)^{2g}/(\mathbb{Z}/2)$, where $\mathbb{Z}/2 = N_{\mathrm{SU}(2)}(S_0^1)/Z(S_0^1)$. Also, $Z_{\mathrm{SU}(2)} = (\mathbb{Z}/2)^{2g}$.

Let S_0^1 also denote the maximal torus in $\mathrm{SO}(3)$ which is the image of S_0^1 under the map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Up to conjugation, the sets of the form $Z(Z(X))$ in $\mathrm{SO}(3)$ are $\mathrm{SO}(3)$, $\mathrm{O}(2) = S_0^1 \cup X_0 S_0^1$ where X_0 is a fixed square root of the identity matrix, the set D of diagonal matrices in $\mathrm{SO}(3)$, S_0^1, $\mathbb{Z}/2 \subset S_0^1$, and $\{I\} = Z(\mathrm{SO}(3))$. Applying Goldman's theorem, we obtain the inclusions

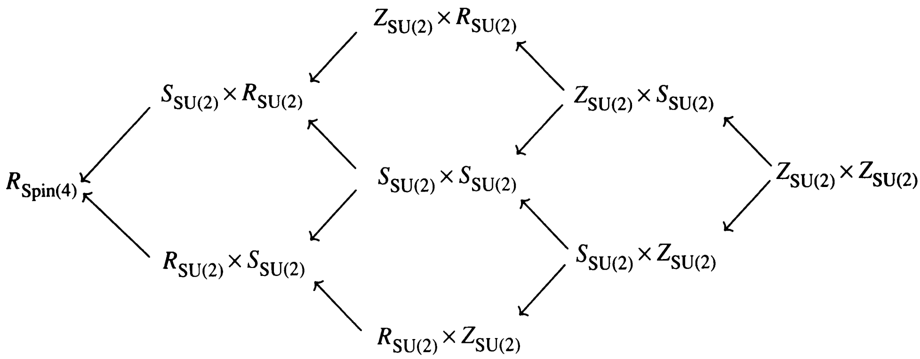
$$\begin{array}{ccccccc}
 & & & A_{\mathrm{SO}(3)} & & & \\
 & & \swarrow & & \nwarrow & & \\
R_{\mathrm{SO}(3)} & \leftarrow & \mathrm{O}_{\mathrm{SO}(3)} & & P_{\mathrm{SO}(3)} & \leftarrow & Z_{\mathrm{SO}(3)}, \\
 & & \nwarrow & & \swarrow & & \\
 & & & S_{\mathrm{SO}(3)} & & &
\end{array}$$

where $\mathrm{O}_{\mathrm{SO}(3)}$ is the set of conjugacy classes of representations in $\mathrm{O}(2)$, $A_{\mathrm{SO}(3)}$ is the set of conjugacy classes of representations in D, $S_{\mathrm{SO}(3)} = (S_0^1)^{2g}/(\mathbb{Z}/2)$ is the set of conjugacy classes of representations in S_0^1, $P_{\mathrm{SO}(3)}$ is the set of conjugacy classes of representations in $\mathbb{Z}/2$, and $Z_{\mathrm{SO}(3)}$ is the set of representations in the center of $\mathrm{SO}(3)$—in this case, the trivial representation. Now $Z(\rho)$ is $\mathbb{Z}/2$ if $\rho \in \mathrm{O}_{\mathrm{SO}(3)} - (A_{\mathrm{SO}(3)} \cup S_{\mathrm{SO}(3)})$ and $\mathbb{Z}/4$ if $\rho \in A_{\mathrm{SO}(3)} - P_{\mathrm{SO}(3)}$. Also $Z(\rho)$ is $\mathrm{O}(2)$ if $\rho \in P_{\mathrm{SO}(3)} - Z_{\mathrm{SO}(3)}$. It follows that the geometric stratification of $R_{\mathrm{SO}(3)}$ is given by $R_{\mathrm{SO}(3)} \supset S_{\mathrm{SO}(3)} \supset Z_{\mathrm{SO}(3)}$.

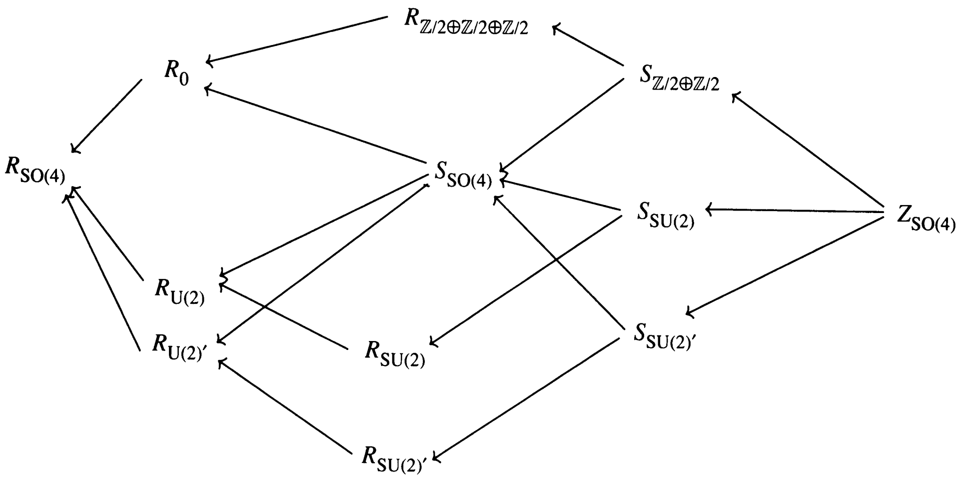
The stratification of $R_{\mathrm{SU}(2)\times S^1} = R_{\mathrm{SU}(2)} \times R_{S^1}$ can be deduced from that of $R_{\mathrm{SU}(2)}$ since S^1 is abelian. We obtain $R_{\mathrm{SU}(2)\times S^1} \supset S_{\mathrm{SU}(2)\times S^1} \supset Z_{\mathrm{SU}(2)\times S^1}$, where again $S_{\mathrm{SU}(2)\times S^1}$ is the set of representations in a fixed maximal torus $S_0^1 \times S^1$ of $\mathrm{SU}(2) \times S^1$ and $Z_{\mathrm{SU}(2)\times S^1}$ is the set of representations in the center of $\mathrm{SU}(2) \times S^1$.

In $\mathrm{U}(2)$, the subgroups of the form $Z(Z(X))$ are $\mathrm{U}(2)$, $S^1 \times S^1$, and $S^1 = Z(\mathrm{U}(2))$. (Note that $Z(\mathrm{SU}(2)) = S^1$ and $Z(\mathrm{O}(2)) = S^1$, so these are not of the form $Z(Z(X))$.) We obtain $R_{\mathrm{U}(2)} \supset S_{\mathrm{U}(2)} \supset Z_{\mathrm{U}(2)}$.

Recalling that $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, we see $R_{\mathrm{Spin}(4)} = R_{\mathrm{SU}(2)} \times R_{\mathrm{SU}(2)}$. Hence the stratification of $R_{\mathrm{Spin}(4)}$ can be deduced from that of $R_{\mathrm{SU}(2)}$. We obtain the diagram of inclusions



In SO(4), the subgroups of the form $Z(Z(X))$ up to conjugation are SO(4), two copies of U(2) (they are not conjugate!), $S(O(2) \times O(2))$, two copies of $SU(2) \supset U(2)$, $S^1 \times S^1$, the set of diagonal matrices $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, two copies of $S^1 \subset Z(U(2))$, and $\mathbb{Z}/2 = Z(SO(4))$. We obtain the diagram of inclusions



As in the SO(3)-case, if we count the dimensions of $Z(\rho)$ for ρ in the various spaces above, we find that R_0, $R_{\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2}$, and $S_{\mathbb{Z}/2\oplus\mathbb{Z}/2}$ are not actual strata. (That is, the dimensions of the Zariski tangent spaces are not reduced.)

Note that the stratifications above induce stratifications of $Q_{j,G}$. For $G =$ SU(2) or SO(3), we have $Q_{j,G} \supset T_{j,G} \supset Q_{j,G} \cap Z_G$ where $T_{j,G} = Q_{j,G} \cap S_G = (S^1_0)^g/(\mathbb{Z}/2)$. Note that $Q_{j,G} \cap Z_G = (Z(G))^g$.

We adopt the following notation: $OQ_{j,SO(3)} = O_{SO(3)} \cap Q_{j,SO(3)}$, $AQ_{j,SO(3)} = A_{SO(3)} \cap Q_{j,SO(3)}$, and $PQ_{j,SO(3)} = P_{SO(3)} \cap Q_{j,SO(3)}$. Moreover, for any representation space X, we use the symbol X^- to denote the representation space X with all subspace(s) shown in the above diagrams removed.

Finally, we study the natural symplectic structures associated with these spaces:

If π is a finitely presented group and G is a Lie group, the Zariski tangent space of $\mathrm{Hom}(\pi, G)$ at ρ can be identified with $Z^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\rho})$, where $\mathfrak{g}$ is the Lie algebra of G, via the map $\rho_t \mapsto u$, where $\rho_t\colon \pi \to G$ is a differentiable 1-parameter family of representations such that $\rho_0 = \rho$ and $\rho_t(x) =$

$\exp(tu(x) + O(t^2))\rho(x)$. Moreover, the tangent space of the Ad-orbit containing ρ is just $B^1(\pi, \mathfrak{g}_{\mathrm{Ad}\,\rho})$ under the above identification, since if $\rho_t(x) = (\exp(tu_0 + O(t^2)))^{-1}\rho(x)(\exp(tu_0 + O(t^2)))$ then the corresponding cocycle is $\mathrm{Ad}\,\rho(x)u_0 - u_0 = \delta u_0$. (For details, see [G1] and [G2].) Thus we define the Zariski tangent space of $\mathrm{Hom}(\pi, G)/G$ at $[\rho]$ to be $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho})$.

The Lie algebra $\mathfrak{g}$ of G affords an Ad-invariant, symmetric, nondegenerate bilinear form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$. On cohomology, B induces a cup product pairing

$$\omega_B\colon H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho}) \to H^2(\pi; \mathbb{R}) \cong \mathbb{R}.$$

We have

Theorem 1.1 (Goldman). *ω_B is a closed, nondegenerate exterior 2-form, making* $\mathrm{Hom}(\pi, G)/G$ *a stratified symplectic space.*

(See [G1] or [G2].)

In fact, since $H^2(\pi_1(H_i); \mathbb{R}) = 0$, ω_B is zero on

$$H^1(\pi_1(H_i); \mathfrak{g}_{\mathrm{Ad}\,\rho}) \subset H^1(\pi_1(\Sigma); \mathfrak{g}_{\mathrm{Ad}\,\rho}).$$

Moreover $\dim(H^1(\pi_1(H_i); \mathfrak{g}_{\mathrm{Ad}\,\rho}) = \frac{1}{2}\dim(H^1(\pi_1(\Sigma); \mathfrak{g}_{\mathrm{Ad}\,\rho})$. Hence $Q_{i,G}$ is lagrangian with respect to ω_B.

We remark that if $\rho \in Z_G$ then $\mathrm{Ad}\,\rho$ is trivial, and $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho}) = H^1(\pi; \mathbb{R}) \otimes \mathfrak{g}$ for $\pi = \pi_1(H_i)$ or $\pi = \pi_1(\Sigma)$. Since M is a $\mathbb{Q}HS$, $H^1(\pi_1(H_1); \mathbb{R})$ and $H^1(\pi_1(H_2); \mathbb{R})$ are transverse in $H^1(\pi_1(\Sigma); \mathbb{R})$. Hence $Q_{1,G}$ and $Q_{2,G}$ are transverse at Z_G.

1.3. **Normal bundles.** We will need a thorough understanding of the normal bundle of the space S_G of abelian representations in R_G for each G.

If $\rho \in S_G^-$, the $\pi_1(\Sigma)$-module $\mathfrak{g}_{\mathrm{Ad}\,\rho}$ decomposes as $\mathfrak{h}_{\mathrm{Ad}\,\rho} \oplus \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho}$, where $\mathfrak{h}_{\mathrm{Ad}\,\rho}$ is the Lie algebra of the fixed maximal torus $T_0 = S^1_0$ or $(S^1 \times S^1)_0$ in G and $\mathfrak{h}^\perp_{\mathrm{Ad}\,\rho}$ is its orthogonal complement with respect to B. Thus

$$H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho}) = H^1(\pi; \mathfrak{h}_{\mathrm{Ad}\,\rho}) \oplus H^1(\pi; \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho})$$

for $\pi = \pi_1(\Sigma)$ or $\pi = \pi_1(H_i)$. Let ν_G be the Zariski normal bundle of S_G^- in R_G, so the fiber at ρ is $H^1(\pi_1(\Sigma); \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho})$. Let $\eta_{j,G}$ be the Zariski normal bundle of $T^-_{j,G}$ in $Q_{j,G}$ at ρ, so the fiber at ρ is $H^1(\pi_1(H_i); \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho})$.

To describe the actual normal bundles, we have the Marsden-Weinstein symplectic quotient:

Theorem 1.2. *Let* $[\rho] \in \mathrm{Hom}(\pi; G)/G$ *and* $x \in H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho})$. *Then x is tangent to a path in* $\mathrm{Hom}(\pi, G)/G$ *if and only if* $[x, x] = 0$. *Hence $[\rho]$ has a neighborhood in* $\mathrm{Hom}(\pi, G)/G$ *diffeomorphic to*

$$\{x \in H^1(\pi; \mathfrak{g}_{\mathrm{Ad}\,\rho}) | [x, x] = 0\}/\,\mathrm{stab}(\rho).$$

A discussion of this can be found in [G1].

It follows that the normal bundle ξ_G of S_G^- in R_G is

$$\xi_G = \{x \in \nu_G | [x, x] = 0\}/T_0$$

while the normal bundle of $T^-_{j,G}$ in $Q_{j,G}$ is

$$\theta_{j,G} = \eta_{j,G}/T_0.$$

For details on the remaining definitions in this chapter, the reader is referred to [W].

If we fix a metric on Σ, take B to be positive definite, and identify $H^1_{\text{de Rham}}(\Sigma; \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$ and $H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$, we get a Hodge star operator

$$*\colon H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}) \to H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$$

and a Hodge metric

$$\langle\ ,\ \rangle\colon H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}) \times H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}) \to \mathbb{R}$$

defined by $\langle \alpha, \beta\rangle = \int_\Sigma B(\alpha, *\beta)$. Then $*$ gives an almost complex structure compatible with ω_B:

$$\omega_B(\alpha, \beta) = -\langle \alpha, *\beta\rangle$$

for each α and β in $H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$.

There is a second almost complex structure on $H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$ as follows: If $G = \mathrm{SU}(2)$, fix a vector $v \in \mathfrak{h}_{\mathrm{Ad}\,\rho}$. There is a map $J\colon \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho} \to \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}$ satisfying $B([a, b], v) = B(a, Jb)$ for all $a, b \in \mathfrak{h}^{\perp}$, where $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie commutator. The action of $\mathrm{Ad}(T_0)$ commutes with J, and $J^2 = -1$ as long as B is scaled properly. Hence $H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$ inherits an almost complex structure from $\mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}$, which we also denote J. Finally, this induces a complex structure on $H^1(\pi_1(\Sigma); \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho})$ for $\mathrm{SO}(3)$, $\mathrm{SU}(2) \times S^1$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$ via the natural maps of Lie algebras. If we lift ν_G to the double cover $\widetilde{S}^-_G$ of S^-_G, then J gives an almost complex structure to the bundle ν_G. From now on, view ν_G and $\eta_{j,G}$ as lifted to $\widetilde{S}^-_G$.

In [W], Walker points out that $\nu_{\mathrm{SU}(2)}$ decomposes as $A^+ \oplus A^-$, where $J*|_{A^\pm} = \pm 1$. The solutions to $[x, x] = 0$ are the cone on the product of the unit sphere bundles of A^+ and A^-. Hence for SU(2)-theory $\xi_\rho \cong C((S^{2g-3} \times S^{2g-3})/S^1)$ and $\theta_{j,\rho} \cong C(\mathbb{C}P^{g-2})$. Clearly this carries over for $\mathrm{SO}(3)$.

Note that $\eta_{j,G}$ is totally real with respect to $*$, yet complex with respect to J. Call such subspaces of ν_G *complex lagrangians.*

Let $\det^1(\nu_G)$ be the determinant line bundle of ν_G over $\widetilde{S}^-_G$. There is a map $\det^1\colon \{\text{complex lagrangians of } \nu_{G,p}\} \to \det^1(\nu_{G,p})$ as follows:

Elements of $\det(\nu_{G,p})$ are represented by bases of $\nu_{G,p}$ over $\mathbb{C}$. Two such bases are identified if the linear map connecting them has determinant 1. Given a complex lagrangian L of ν_G, choose a basis (over $\mathbb{R}$) of L which is oriented with respect to the orientation coming from the J-complex structure. Since L is totally real with respect to $*$, any two such bases differ by an element of $\mathrm{GL}(2g-2, \mathbb{R}) \subset \mathrm{GL}(2g-2, \mathbb{C})$ (or $\mathrm{GL}(4g-4, \mathbb{R}) \subset \mathrm{GL}(4g-4, \mathbb{C})$), which is well defined up to conjugacy. Thus the basis is a well-defined element of $\det(\nu_{G,p})$.

We have the following:

Theorem 1.3. (a) *$\det(\nu_G)$ extends naturally over all of $\widetilde{S}_G$, and the fiber over $\rho \in Z_G$ is naturally identified with* $\det^1(H^1(\Sigma; \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}))$.

(b) *$\eta_{j,G}$ gives rise to a section $\det^1(\eta_{j,G})$ of $\det^1(\nu_G)$ over $\widetilde{T}^-_{j,G}$ which extends continuously to $\det^1(H^1(H_j; \mathfrak{h}^{\perp}_{\mathrm{Ad}\,\rho}))$ over $\rho \in \widetilde{T}^-_{j,G} \cap Z_G$.*

Proof. (a) Identify $\widetilde{S}_G$ with the space of flat connections modulo gauge equivalence on the principal S^1_0-bundle over Σ, or equivalently on $\Sigma \times \mathfrak{h}^\perp$, where the two bundles are identified via the representation $\mathrm{Ad}(S^1_0)$. Following Quillen [Q], $\det(\nu_G)$ may be identified with the determinant bundle $\det(D)$ of a family of Fredholm operators. Details for $\mathrm{SU}(2)$ are given in [W]; the argument for the other groups is similar. $\det(D)$ is a bundle over all of $\widetilde{S}_G$, and the fiber at $p \in Z_G$ is canonically identified with $\det(H^1(\Sigma; \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho}))$.

(b) In [W], Walker identifies $\det(\nu_{\mathrm{SU}(2)})$ in a neighborhood of the identity with the determinant of another family of Fredholm operators $\det(D'_\rho)$. This bundle has a smooth section near the identity which agrees with $\det^1(\eta_{j,\mathrm{SU}(2)})$ for $\rho \neq 1$ and coincides with $\det^1(H^1(H_1; \mathfrak{h}^\perp_{\mathrm{Ad}\,\rho}))$ at $\rho = 1$. The argument is similar for the other handlebody H_2 and for other points in $Z_{\mathrm{SU}(2)}$. The argument for the other groups is similar. □

1.4. **Orientation conventions.** In this section we establish orientation conventions to be used for the remainder of the paper. We remark that these agree with those used by Walker in the development of the $\mathrm{SU}(2)$-invariant.

Write $[Y]$ to denote the orientation of a manifold Y. By the orientation of a bundle, we mean an orientation of the fibers.

Orient Σ so that $[M]$ is the orientation of $[\Sigma]$ followed by a normal vector pointing into H_2. This gives an identification of $H^2(\Sigma; \mathbb{R})$ with $\mathbb{R}$, thereby fixing the signs of ω_B.

The spaces R_G, S_G, ν_G, and ξ_G are symplectic vector spaces, so they inherit an orientation from ω_B.

The bundles $\eta_{j,G}$ and $\theta_{j,G}$ lifted to $\widetilde{T}_{j,G}$ have J-complex structures; this determines their orientation.

Choose orientations of $\widetilde{T}_G$ so that $[\widetilde{S}_G] = [\widetilde{T}_{1,G}][\widetilde{T}_{2,G}]$, where $[\widetilde{S}_G]$ is the orientation lifted from S_G.

Given a fibering $Y \to E \to B$, choose orientations so that $[E] = [B][Y]$. Regard spaces of unit vectors as quotients of spaces of nonunit vectors by $\mathbb{R}^+$. Let $[G]$, $[S^1_0]$, $[(S^1 \times S^1)_0]$ (the fixed maximal torus in $\mathrm{SU}(2) \times S^1$, $\mathrm{U}(2)$, $\mathrm{Spin}(4)$, and $\mathrm{SO}(4)$), and $[\mathbb{R}^+]$ be the standard orientations. Define

$$[R_G] = [S_G][\xi_G], \quad [Q_{j,G}] = [\widetilde{T}_{j,G}][\theta_{j,G}], \quad [\eta_{j,G}] = [\theta_{j,G}][T_0].$$

(Here, note that $[\widetilde{T}_{j,G}][\theta_{j,G}]$ determines an orientation of a cover of a neighborhood of $T^-_{j,G}$ in $Q_{j,G}$, so the second equation makes sense.)

Orient boundary components individually according to the "inward-pointing normal last" convention.

Finally, if A and B are oriented, properly embedded submanifolds of an oriented manifold Y intersecting transversely at X, orient the normal bundle ζ^Y_A of A in Y such that $[Y] = [\zeta^Y_A][A]$, and orient X such that $[B] = [\zeta^B_X][X]$, where $\zeta^B_X = \zeta^Y_{A|X}$ is the normal bundle of X in B.

2. Definition of the $\mathrm{SO}(3)$-invariant

2.1. **The maps $R_{\mathrm{Spin}(n)} \to R_{\mathrm{SO}(n)}$ and $\det^1(\nu_{\mathrm{SU}(2)}) \to \det^1(\nu_{\mathrm{SO}(3)})$.** Note that the map $\mathrm{Spin}(n) \to \mathrm{SO}(n)$ induces a homomorphism $R^{*\#}_{\mathrm{Spin}(n)} \to R^{*\#}_{\mathrm{SO}(n)}$. Since $\pi_1(\Sigma^*)$ is a free group, this map is surjective. Moreover, the conjugation action of $\operatorname{Spin}(n)$ on itself descends to a faithful action of $SO(n)$ on $\operatorname{Spin}(n)$. Hence we in fact get a surjection $R^*_{\operatorname{Spin}(n)} \to R^*_{SO(n)}$. Similarly, we get surjections $Q_{i,\operatorname{Spin}(n)} \to Q_{i,SO(n)}$ and maps $R_{\operatorname{Spin}(n)} \to R_{SO(n)}$ and $Q_{1,\operatorname{Spin}(n)} \cap Q_{2,\operatorname{Spin}(n)} \to Q_{1,SO(n)} \cap Q_{2,SO(n)}$. These latter maps are *not* surjections in general. In particular, if $H^2(M;\mathbb{Z}/2) \neq 0$, then not all homomorphisms $\pi_1(M) \to SO(n)$ factor through $\operatorname{Spin}(n)$. This section is devoted to finding the inverse images of $Q^\#_{1,SO(n)} \cap Q^\#_{2,SO(n)}$ in $R^{*\#}_{\operatorname{Spin}(n)}$. For $n = 3$ we in fact count the inverse images of $Q_{1,SO(n)} \cap Q_{2,SO(n)}$ in $R^*_{\operatorname{Spin}(n)}$.

We begin with the map $R^\#_{\operatorname{Spin}(n)} \to R^\#_{SO(n)}$. The existence of a factorization of a homomorphism $\rho\colon \pi_1(M) \to SO(n)$ through $\operatorname{Spin}(n)$ is equivalent to the existence of a spin structure on the associated flat $SO(n)$-bundle ξ_ρ. (Recall that a spin structure on ξ is a cohomology class in $H^1(E(\xi);\mathbb{Z}/2)$ whose restriction to each fiber is a generator of the cyclic group $H^1(\text{fiber};\mathbb{Z}/2)$. See [M].) An $SO(n)$-bundle ξ can be given a spin structure if and only if $\omega_2(\xi) = 0$. Furthermore, if $\omega_2(\xi) = 0$, the number of distinct spin structures is $|H^1(B(\xi);\mathbb{Z}/2)|$. We obtain

Lemma 2.1. *$R^\#_{\operatorname{Spin}(n)}$ is a $|H^1(\Sigma;\mathbb{Z}/2)|$-to-1 cover of its image in $R^\#_{SO(n)}$. $Q^\#_{i,\operatorname{Spin}(n)}$ is a $|H^1(H_i;\mathbb{Z}/2)|$-to-1 cover of $Q^\#_{i,SO(n)}$. $Q^\#_{1,\operatorname{Spin}(n)} \cap Q^\#_{2,\operatorname{Spin}(n)}$ is a $|H^1(M;\mathbb{Z}/2)|$-to-1 cover of its image in $Q^\#_{1,SO(n)} \cap Q^\#_{2,SO(n)}$.*

From now on we write $\omega_2(\rho)$ instead of $\omega_2(\xi_\rho)$. We write $R^\#_\alpha(M)$ for $\{\rho \in Q^\#_{1,SO(n)} \cap Q^\#_{2,SO(n)} | \omega_2(\rho) = \alpha\}$.

We now investigate the sets $R^\#_\alpha(M)$ for $\alpha \neq 0$.

Consider the Mayer-Vietoris sequence

$$\cdots \to H^1(\Sigma;\mathbb{Z}/2) \to H^2(M;\mathbb{Z}/2) \to H^2(H_1;\mathbb{Z}/2) \oplus H^2(H_2;\mathbb{Z}/2) \to \cdots.$$

Note that $H^2(H_i;\mathbb{Z}/2) = 0$, so the map $H^1(\Sigma;\mathbb{Z}/2) \to H^2(M;\mathbb{Z}/2)$ is surjective. Moreover, $H^1(\Sigma;\mathbb{Z}/2) \cong \operatorname{Hom}(H_1(\Sigma), \mathbb{Z}/2) \cong \operatorname{Hom}(\pi_1(\Sigma), \mathbb{Z}/2)$ by the Universal Coefficient Theorem. We show

Lemma 2.2. *Let $\alpha \in H^2(M;\mathbb{Z}/2)$, and let β_α be an inverse image of α in $H^1(\Sigma;\mathbb{Z}/2)$. Let $\imath_{\beta_\alpha}$ be the involution of $R^\#_{\operatorname{Spin}(n)}$ induced by the homomorphism $\beta_\alpha \in H^1(\Sigma;\mathbb{Z}/2) = \operatorname{Hom}(\pi_1(\Sigma), \mathbb{Z}/2)$. (That is, $\imath_{\beta_\alpha}(\rho) = \rho \otimes \beta_\alpha$.) Let $\rho \in R^\#_\alpha(M)$. Then ρ has an inverse image in $Q^\#_{1,\operatorname{Spin}(n)} \cap \imath_{\beta_\alpha} Q^\#_{2,\operatorname{Spin}(n)}$.*

Proof. We begin by noting that $\rho_{|H_i}$ has a lift to $\operatorname{Spin}(n)$ for $i = 1, 2$ since $H^2(H_i;\mathbb{Z}/2) = 0$. The lifts do not agree on Σ since $\omega_2(\rho) \neq 0$; however they agree up to sign. The cohomology class α is the obstruction to extending a lift to the generators of $\pi_1(M)$ which are generators of $H_1(M;\mathbb{Z}/2)$, where $H_1(M;\mathbb{Z}/2)$ is identified with $H^2(M;\mathbb{Z}/2)$ via Poincaré duality. In other words, if the ith entry of α is 0, we may choose lifts of $\rho_{|H_1}$ and $\rho_{|H_2}$ which agree on the ith generators; otherwise not. Thus, if we change the sign of the lift of $\rho_{|H_2}$ in accordance with β_α, it will agree with a lift of $\rho_{|H_1}$. Hence, ρ has a lift to $Q^\#_{1,\operatorname{Spin}(n)} \cap \imath_{\beta_\alpha} Q^\#_{2,\operatorname{Spin}(n)} \subset R^\#_{\operatorname{Spin}(n)}$. □

Proposition 2.3. *For each β_α in $H^1(\Sigma;\mathbb{Z}/2)$, the subspace*

$$Q^\#_{1,\operatorname{Spin}(n)} \cap \imath_{\beta_\alpha} Q^\#_{2,\operatorname{Spin}(n)}$$

of $R^\#_{\operatorname{Spin}(n)}$ is a $|H^1(M;\mathbb{Z}/2)|$-to-1 cover of $R^\#_\alpha(M)$ in $R^\#_{SO(n)}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
0 & & H^2(\Sigma;\mathbb{Z}/2) \\
\uparrow & & \uparrow \\
H^1(\mathrm{SO}(n);\mathbb{Z}/2)\oplus H^1(\mathrm{SO}(n);Z/2) & \longrightarrow & H^1(\mathrm{SO}(n);\mathbb{Z}/2) \\
\uparrow & & \uparrow \\
H^1(E(\xi_{\rho|_{H_1}});\mathbb{Z}/2)\oplus H^1(E(\xi_{\rho|_{H_2}});\mathbb{Z}/2) & \longrightarrow & H^1(E(\xi_{\rho|_\Sigma});\mathbb{Z}/2) \\
\uparrow & & \uparrow \\
0\to H^1(M;\mathbb{Z}/2)\to H^1(H_1;\mathbb{Z}/2)\oplus H^1(H_2;Z/2) & \longrightarrow & H^1(\Sigma;\mathbb{Z}/2)\to H^2(M;\mathbb{Z}/2)\to 0 \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

Here, ξ_ρ is the principal $\mathrm{SO}(n)$-bundle with flat connection corresponding to ρ. The vertical sequences are extracted from the spectral sequences of the fibration ξ_ρ. (See [M].) The map $H^1(\mathrm{SO}(n);\mathbb{Z}/2)\oplus H^1(\mathrm{SO}(n);\mathbb{Z}/2)\to H^1(\mathrm{SO}(n);\mathbb{Z}/2)$ is the map

$$(1,0)\mapsto 1,\qquad (0,1)\mapsto 1,$$

from $\mathbb{Z}/2\oplus\mathbb{Z}/2\to\mathbb{Z}/2$.

A lift of $\rho|_{H_1}$ is an element of $H^1(E(\xi_{\rho|_{H_1}}))$ mapping to 1 in $H^1(\mathrm{SO}(n);\mathbb{Z}/2)$. Each map $H^1(E(\xi_{\rho|_{H_1}}))\to H^1(E(\xi_{\rho|_\Sigma}))$ takes such a lift to the corresponding lift of $\rho|_\Sigma$; hence the map $H^1(E(\xi_{\rho|_{H_1}}))\oplus H^1(E(\xi_{\rho|_{H_2}}))\to H^1(E(\xi_{\rho|_\Sigma}))$ takes a pair (s_1, s_2) of such lifts to their difference in $H^1(E(\xi_{\rho|_\Sigma}))$. Thus, an element of $Q^\#_{1,\mathrm{Spin}(n)}\cap\imath_{\beta_\alpha}Q^\#_{2,\mathrm{Spin}(n)}$ is a pair (s_1, s_2) in $H^1(E(\xi_{\rho|_{H_1}}))\oplus H^1(E(\xi_{\rho_{|_{H_2}}}))$ which maps to $(1, 1)$ in $H^1(\mathrm{SO}(n);\mathbb{Z}/2)\oplus H^1(\mathrm{SO}(n);\mathbb{Z}/2)$ and maps to the image of β_α in $H^1(E(\xi_{\rho_{|_\Sigma}}))$. The number of pairs (s_1, s_2) in $H^1(E(\xi_{\rho|_{H_1}}))\oplus H^1(E(\xi_{\rho_{|_{H_2}}}))$ mapping to $(1, 1)$ in $H^1(\mathrm{SO}(n);\mathbb{Z}/2)\oplus H^1(\mathrm{SO}(n);/\mathbb{Z}/2)$ is equal to the cardinality of $H^1(H_1;\mathbb{Z}/2)\oplus H^1(H_2;\mathbb{Z}/2)$. The number of these which map to a fixed element x in the image of $H^1(E(\xi_{\rho_{|_{H_1}}}))\oplus H^1(E(\xi_{\rho_{|_{H_2}}}))$ in $H^1(E(\xi_{\rho|_\Sigma}))$ is equal to the cardinality of the kernel of the map

$$H^1(H_1;\mathbb{Z}/2)\oplus H^1(H_2;\mathbb{Z}/2)\to H^1(\Sigma;\mathbb{Z}/2);$$

that is, $|H^1(M;\mathbb{Z}/2)|$. □

We now restrict our attention to the case $n=3$. Recall $\mathrm{Spin}(3)=\mathrm{SU}(2)$. We know that each $\rho\in Q^\#_{1,\mathrm{SO}(3)}\cap Q^\#_{2,\mathrm{SO}(3)}$ has $|H^1(M;\mathbb{Z}/2)|$ lifts to $Q^\#_{1,\mathrm{SU}(2)}\cap\imath_{\beta_\alpha}Q^\#_{2,\mathrm{SU}(2)}$. Moreover, conjugate representations in $\mathrm{SU}(2)$ map to conjugate representations in $\mathrm{SO}(3)$, so the lifts of elements of $Q_{1,\mathrm{SO}(3)}\cap Q_{2,\mathrm{SO}(3)}$ are well defined. It remains to be seen when distinct lifts of an element of $Q_{1,\mathrm{SO}(3)}\cap Q_{2,\mathrm{SO}(3)}$ are conjugate. We show

Proposition 2.4. *Let n_ρ be the number of components of $Z(\rho)$ in* SO(3). *Then* $[\rho] \in Q_{1,\mathrm{SO}(3)} \cap Q_{2,\mathrm{SO}(3)}$ *has* $|H^1(M;\mathbb{Z}/2)|/n_\rho$ *lifts in* $Q_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q_{2,\mathrm{SU}(2)}$.

Proof. We begin by noting that the map $R_{\mathrm{SU}(2)} \to R_{\mathrm{SO}(3)}$ preserves the dimension of the Zariski tangent space at each point. Let $\rho \in Q^{\#}_{1,\mathrm{SO}(3)} \cap Q^{\#}_{2,\mathrm{SO}(3)}$, and let $\overline{\rho}$ be a lift in $Q^{\#}_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q^{\#}_{2,\mathrm{SU}(2)}$. Then $Z(\rho)$ is the image of the set $\{X \in \mathrm{SU}(2) | \exists x \in \mathrm{Hom}(\pi_1(\Sigma), \mathbb{Z}/2) \text{ with } X \cdot \overline{\rho} = \overline{\rho} \otimes x\}$ in SO(3). But the set of $X \in \mathrm{SU}(2)$ such that $X \cdot \overline{\rho} = \overline{\rho} \otimes x$ is just $Z(\overline{\rho})X_0$, where X_0 is a fixed solution. It follows that the dimension of $Z(\rho)$ is equal to the dimension of a finite number of copies of $Z(\overline{\rho})$; i.e. $\dim Z(\overline{\rho})$. Now recall that the dimension of the Zariski tangent space at a point $[\rho]$ in R_G is determined by the dimension of G and the dimension of $Z(\rho)$. (See [G].)

Let $\rho \in Q_{1,\mathrm{SO}(3)} \cap Q_{2,\mathrm{SO}(3)}$ with $\dim Z(\rho) = 0$. If there exist distinct conjugate lifts of ρ in $Q_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q_{2,\mathrm{SU}(2)}$, then there exists a nontrivial element of SO(3) in $Z(\rho)$. It follows that $\rho(\pi_1(\Sigma)) \subseteq Z(Z(\rho))$ is strictly contained in SO(3), and hence $\rho(\pi_1(\Sigma)) \subseteq \mathrm{O}(2)$. Similarly, if there exist at least three distinct conjugate lifts of ρ in $Q_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q_{2,\mathrm{SU}(2)}$, then $\rho(\pi_1(\Sigma)) \subseteq Z(Z(\rho))$ is contained in D.

Now assume $\rho(\pi_1(\Sigma)) \subseteq \mathrm{O}(2)$ and $\rho \notin A_{\mathrm{SO}(3)} \cup S_{\mathrm{SO}(3)}$, and let $\overline{\rho}$ be a lift of ρ in $Q_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q_{2,\mathrm{SU}(2)}$. Then $Z(\rho) = \mathbb{Z}/2$, and also $Z(\rho_i) = \mathbb{Z}/2$, where ρ_i is just ρ viewed as an element of $\mathrm{Hom}(\pi_1(H_i), \mathrm{O}(2))$. Then there exists $X_i \in \mathrm{SU}(2)$, $X_i \neq \pm I$ with $X_i \cdot \overline{\rho}_i = \overline{\rho}_i \otimes x_i$ for some nontrivial $x_i \in \mathrm{Hom}(\pi_1(H_i), \mathbb{Z}/2)$. This yields a lift of ρ in $Q_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q_{2,\mathrm{SU}(2)}$ distinct from yet conjugate to $\overline{\rho}$, namely $\overline{\rho} \otimes x$, where $x = \mathrm{im}(x_1) = \imath_{\beta_\alpha} \mathrm{im}(x_2)$ in $\mathrm{Hom}(\pi_1(\Sigma_i), \mathrm{O}(2))$. Similarly, if $\rho \in A^-_{\mathrm{SO}(3)}$, then ρ has four distinct conjugate lifts in $Q^{\#}_{1,\mathrm{SU}(2)} \cap \imath_{\beta_\alpha} Q^{\#}_{2,\mathrm{SU}(2)}$.

Finally, similar arguments using S^1 in place of SU(2) show that a representation with 1-dimensional centralizer has multiple distinct but conjugate lifts if and only if its image is contained in (and hence equal to) $\mathbb{Z}/2$. In this case, there are exactly 2 distinct conjugate lifts of the representation. □

To conclude this section, we show

Lemma 2.5. $\det^1(\nu_{\mathrm{SU}(2)})$ *is the induced bundle* $f^*\det^1(\nu_{\mathrm{SO}(3)})$ *where* $f: \widetilde{S}_{\mathrm{SU}(2)} \to \widetilde{S}_{\mathrm{SO}(3)}$ *is the map of degree* 2 *which is the restriction to* S^1_0 *of the map* $R_{\mathrm{SU}(2)} \to R_{\mathrm{SO}(3)}$. *Moreover* $\det^1(\eta_{j,\mathrm{SU}(2)}) = f^*\det^1(\eta_{j,\mathrm{SO}(3)})$.

Proof. This is clear if we restrict our attention to bundles over $\widetilde{S}^-_G$. Moreover, if $\det(D_G)$ denotes the determinant line bundle of the family of Fredholm operators used to extend $\det(\nu_G)$ over $\widetilde{S}_G$, it is clear that $\det(D_{\mathrm{SU}(2)}) = f^*\det(D_{\mathrm{SO}(3)})$. (View f as the corresponding map of spaces of flat connections on the trivial principal S^1_0-bundle modulo gauge equivalence. Again, see [Q and W] for details on the definition of D.) Similarly, the construction of the extension of $\det^1(\eta_{j,G})$ over $T_{j,G} - Z_G$ to a section over $T_{j,G}$ clearly agrees for $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, so $\det^1(\eta_{j,\mathrm{SU}(2)}) = f^*\det^1(\eta_{j,\mathrm{SO}(3)})$. □

We also obtain

Corollary 2.6. *The first Chern class* $c_1(\det^1(\nu_{\mathrm{SO}(3)}))$ *of* $\det^1(\nu_{\mathrm{SO}(3)})$ *is represented by a multiple* $\omega_{\mathrm{SO}(3)}$ *of* ω_B. *If* T^2 *is a symplectic* 2*-torus in* $\widetilde{S}_{\mathrm{SO}(3)}$ *corresponding to a* 2*-dimensional symplectic summand of* $H^1(\Sigma;\mathbb{Z})$, *then* $\int_{T^2} \omega_{\mathrm{SO}(3)} = -2$.

Proof. This follows from Lemma 2.5 and Proposition 1.14 of [W]. □

2.2. **The** SO(3)**-invariant.** We now define the invariant $\lambda_{\mathrm{SO}(3)}(M)$, where M is a $\mathbb{Q}HS$. The definition is similar to that of $\lambda_{\mathrm{SU}(2)}(M)$.

Let $G = \mathrm{SU}(2)$, $\mathrm{SO}(3)$, $\mathrm{SU}(2) \times S^1$, or $\mathrm{U}(2)$.

Definition 1. An isotopy $\{h_t\}_{0\leq t\leq 1}$ of R_G is *special* if
1. $h_t(Q_{1,G})$ is transverse to $Q_{2,G}$ at $Q_{1,G} \cap Q_{2,G} \cap Z_G$ for all t;
2. $h_{t|_{S_G}} = $ identity for all t;
3. $h_{t*}\colon TR_{G|_{S_G^-}} \to TR_{G|_{S_G^-}}$ is symplectic, thus preserving the fibers of the normal bundle ν_G.

Henceforth, all representation spaces are taken to consist of SO(3)-representations unless otherwise indicated; we omit the subscripts.

Standard arguments show that Q_1 and Q_2 can be put into general position by a special isotopy. (See for example, [W, Proposition 1.19].) Assume henceforth that Q_1 and Q_2 are in general position. We give numbers $s(p)$ for $p \in OQ_1^- \cap OQ_2^-$ and $AQ_1^- \cap AQ_2^-$ and numbers $I(p)$ for $p \in T_1 \cap T_2$ such that the sum

$$\sum_{p\in Q_1^-\cap Q_2^-} \operatorname{sign}(p) + \sum_{p\in (OQ_1\cap OQ_2)-S} s(p) + \sum_{p\in T_1\cap T_2} I(p)$$

is independent of the choice of special isotopy, the choice of orientations of $\widetilde{T}_1$ and $\widetilde{T}_2$, and the choice of Heegaard decomposition of M.

Let $p \in (OQ_1 \cap OQ_2) - S$. If $p \notin A$ (resp. $p \in A$), let U be a neighborhood of p in R such that each point in U has at most two (respectively, four) inverse images in each component of $\phi^{-1}(U)$, where $\phi\colon R_{\mathrm{SU}(2)} \to R_{\mathrm{SO}(3)}$. Let $\widetilde{U}$ be the double (respectively, four-fold) cover of U, and let $\widetilde{Q}_i \cap \widetilde{U}$ be the lift of $Q_i \cap U$ to $\widetilde{U}$. Note that the orientations of $\widetilde{U}$ and $\widetilde{Q}_i \cap \widetilde{U}$ are well defined, induced by the lifts of the orientations of $U - O$ and $Q_i \cap U - O$. Let $\tilde{p}$ be the lift of p to $\widetilde{U}$. Define

$$s(p) = \begin{cases} \frac{1}{2}\operatorname{sign}(\tilde{p}) & \text{if } p \notin A, \\ \frac{1}{4}\operatorname{sign}(\tilde{p}) & \text{if } p \in A. \end{cases}$$

For $p \in T_1 \cap T_2$, we define $I(p)$ as follows:

Let p' and p'' be the inverse images of p in $\widetilde{S}$. (If $p \in Z$, then $p' = p''$.) Choose arcs α'_j from 1 to p' in $\widetilde{T}_j$. Let $\gamma' = \alpha'_1 * (-\alpha'_2)$, and let $\gamma'' = \tau(\alpha'_1) * (-\tau(\alpha'_2))$, where τ is the covering involution of $\widetilde{S}$. Since τ induces multiplication by -1 on $H_1(\widetilde{S};\mathbb{Z})$, γ' is homologous to $-\gamma''$, and we may choose a surface E in $\widetilde{S}$ with $\partial E = \gamma' \cup \gamma''$.

Next we define a trivialization Φ of $\det^1(\nu)$ over ∂E. To do so, we need to define two paths connecting transverse, oriented lagrangian subspaces in a symplectic vector space.

Let L_1 and L_2 be transverse, oriented lagrangian subspaces of a symplectic vector space. Let $e_1, \dots, e_n$ be an oriented basis of L_1. Then there is a unique basis $f_1, \ldots, f_n$ of L_2 such that $\omega(e_i, f_j) = \delta_{i,j}$, where ω is the symplectic structure of the vector space. Set

$$P_\pm(L_1, L_2)_t = \text{oriented span of } g_{1,t}, \ldots, g_{n,t},$$

where

$$g_{i,t} = \cos\left(\frac{\pi}{2}(t-1)\right) e_i \pm \sin\left(\frac{\pi}{2}(t-1)\right) f_i.$$

Note that $P_\pm(L_1, L_2)$ depends continuously on the choice of oriented basis of L_1. Since the set of all such bases is connected, the homotopy class of $P_\pm(L_1, L_2)$ is well defined.

Using these paths, we can define trivializations of $\det^1(\nu)$ over γ' using the map $S^1 \to \det^1(\nu)_{|\gamma'}$ given by

$$\det{}^1(\eta_1)_{|\alpha_1'} * \det{}^1(P_\pm(\eta_{1,p'}, \eta_{2,p'})) * (-\det{}^1(\eta_2)_{|\tau(\alpha_1')}) * (-\det{}^1 P_\pm(\eta_{1,1}, \eta_{2,1})).$$

Define trivializations over γ'' similarly. Finally, let $\Phi_\pm$ be the corresponding trivializations over ∂E, and let Φ be the average of Φ_+ and Φ_-. (That is, if f is an affine function from the set of trivializations to $\mathbb{R}$, then $f(\Phi) = \frac{1}{2}(f(\Phi_+) + f(\Phi_-))$.) Now define

$$I(p) = \begin{cases} \frac{1}{2}\left(c_1(\det^1(\nu_{SO(3)|_E}, \Phi)) - \int_E \omega_{SO(3)}\right) & \text{if } p \notin P, \\ \frac{1}{4}\left(c_1(\det^1(\nu_{SO(3)|_E}, \Phi)) - \int_E \omega_{SO(3)}\right) & \text{if } p \in P^-, \\ 0 & \text{if } p \in Z. \end{cases}$$

Note that if we choose another surface E' with $\partial E' = \gamma' \cup \gamma''$ and define $I'(p) = \frac{1}{2}(c_1(\det^1(\nu_{|E'}), \Phi) - \int_{E'} \omega)$, then $I(p) - I'(p) = 0$, since $E \cup E'$ is closed and ω represents $c_1(\det^1(\nu))$ by Corollary 2.6. Thus $I(p)$ is independent of the choice of spanning surface E.

If we choose another arc α_3' from 1 to p' in $\widetilde{T}_1$, we may choose a surface D lying entirely in $\widetilde{T}_1$ with $\partial D = (-\alpha_3') * \alpha_1' \cup (-\tau(\alpha_3')) * \tau(\alpha_1')$. The trivialization given by $\det^1(\eta_1)$ extends over D, and hence $c_1(\det^1(\nu_{|D\cup E})) = c_1(\det^1(\nu_{|E}))$ and $\int_{D\cup E} \omega = \int_E \omega$. It follows that $I(p)$ is independent of the choice of arc α_1'. The argument that $I(p)$ is independent of the choice of α_2' is similar.

Finally, note that $\int_E \omega$ is independent of the metric on F. Moreover, $c_1(\det^1(\nu))$ depends continuously on the choice of metric and takes on discrete values. Since the set of metrics on F is connected, it follows that $c_1(\det^1(\nu))$ is also independent of the choice of metric, and hence $I(p)$ is.

We define

$$\lambda_{SO(3)}(M) = \frac{\sum_{p \in Q_1^- \cap Q_2^-} \operatorname{sign}(p) + \sum_{p \in (OQ_1 \cap OQ_2) - S} s(p) + \sum_{p \in T_1 \cap T_2} I(p)}{|H_1(M; \mathbb{Z})|}.$$

To see that $\lambda_{SO(3)}$ is well defined, we must show that the right-hand side is independent of the choice of isotopy used to put Q_1 and Q_2 into general position, the choice of orientations of $\widetilde{T}_1$ and $\widetilde{T}_2$, and the choice of Heegaard decomposition. The proof of special isotopy invariance is a slight generalization of Walker's proof for the SU(2)-case. The other proofs are completely analogous to the SU(2)-case, so we omit them.

We first show that $\lambda_{SO(3)}(M)$ is independent of the choice of special isotopy. This is equivalent to showing that given a special isotopy $h\colon R \times I \to R$ with $h(\cdot, 0) = \text{identity}$ such that $Q_1^\dagger = h(Q_1, 1)$ is in general position with Q_2, we have

$$\sum_{p\in Q_1^-\cap Q_2^-} \operatorname{sign}(p) + \sum_{p\in(OQ_1\cap OQ_2)-S} s(p) + \sum_{p\in T_1\cap T_2} I(p)$$
$$= \sum_{p\in Q_1^{\dagger,-}\cap Q_2^-} \operatorname{sign}(p) + \sum_{p\in(OQ_1^\dagger\cap OQ_2)-S} s(p) + \sum_{p\in T_1\cap T_2} I^\dagger(p)$$

where $I^\dagger(p)$ is computed using $Q_1^\dagger$, $\eta_1^\dagger$, etc.

We begin by "unconing" the normal bundle of S^- in R. More precisely:

Let $\hat{\xi}$ be the unit normal bundle of S^- in R. For some neighborhood N of $\hat{\xi}\times\{0\}$, the map $\hat{\xi}+I \xrightarrow{e} R$ given by $(x, t)\mapsto \exp(tx)$ is a diffeomorphism onto its image. Let $\overline{R}^- = R^- \cup_e N$. Define $\overline{Q_j^-} \subset \overline{R}^-$ similarly.

Now $\overline{Q_j^-}$ and $\overline{R}^-$ are quotients of manifolds with boundary by a finite group action. Hence there is a well-defined intersection theory for $\overline{Q_1^-}$ and $\overline{Q_2^-}$ in $\overline{R}^-$. The definition of $s(p)$ for p in $OQ_1^-\cap OQ_2^-$ or $AQ_1^-\cap AQ_2^-$ given in the previous section is precisely that needed for such an intersection theory.

Abusing notation, we also let h denote the map $\overline{R}^-\times I\to\overline{R}^-$ induced by $h\colon R\times I\to R$. We may assume $\overline{Q_1^-}\times I$ and $\overline{Q_2^-}$ are in general position in $\overline{R}$. We obtain

$$\sum_{p\in Q_1^-\cap Q_2^-} \operatorname{sign}(p) + \sum_{p\in(OQ_1\cap OQ_2)-S} s(p) - \sum_{p\in Q_1^{\dagger,-}\cap Q_2^-} \operatorname{sign}(p) - \sum_{p\in(OQ_1^\dagger\cap OQ_2)-S} s(p)$$
$$= -\sum_{c\in h(\hat{\theta}_1\times I)\cap\hat{\theta}_2} n(c)$$

where $\hat{\theta}_i$ is the unit normal bundle of T_i^- in Q_i and $n(c)$ is defined as follows:

Orient $\overline{Q_1^-}\times I$ as a product. Orient $h(\overline{Q_1^-}\times I)\cap\overline{Q_2^-}$ as an intersection. View $h(\hat{\theta}_1\times I)\cap\hat{\theta}_2$ as a subset of $\partial(h(\overline{Q_1^-}\times I\cap\overline{Q_2^-})$, and define $n(c)$ for $c\in h(\hat{\theta}_1\times I)\cap\hat{\theta}_2$ to be 1 (resp. 1/2, resp. 1/4) if the inward pointing normal at c is positively oriented and $c\notin O$ (resp. $c\in O$, $c\notin A$, resp. $c\in A$) and -1 (resp. $-1/2$, resp. $-1/4$) otherwise.

Thus we must show

$$\sum_{p\in T_1\cap T_2} (I^\dagger(p) - I(p)) = \sum_{c\in(\hat{\theta}_1\times I)\cap\hat{\theta}_2} n(c).$$

Now since special isotopies are fiber-preserving, we have

$$(\hat{\theta}_1\times I)\cap\hat{\theta}_2 = \bigcup_{p\in T_1\cap T_2} (\hat{\theta}_{1,p}\times I)\cap\hat{\theta}_{2,p}.$$

Thus it is sufficient to show for each $p\in T_1\cap T_2$,

$$I(p) - I^\dagger(p) = \sum_{c\in(\hat{\theta}_{1,p}\times I)\cap\hat{\theta}_{2,p}} n(c).$$

Note that this statement is trivial if $p \in Z$, since special isotopies keep Q_1 and Q_2 Zariski transverse at Z.

In [W], Walker applies the orientation conventions of §1.5 to get

$$\operatorname{sign}(n(c)) = (-1)^{g-1}\frac{[\hat{\theta}_1][I][\hat{\theta}_2]}{[\hat{\xi}]}.$$

If $p \in T_1^- \cap T_2^-$, this tells us

$$\sum_{c\in(\hat{\theta}_{1,p}\times I)\cap\hat{\theta}_{2,p}} n(c) = (-1)^{g-1}\langle(\hat{\theta}_{1,p'}\times I),\ \hat{\theta}_{2,p'}\rangle_{\hat{\xi}_{p'}}$$

where p' is a lift of p to $\widetilde{S}$, the double cover of S. If $p \in P$ and p' is the unique lift of p to $\widetilde{S}$, note that the fiber $\hat{\xi}_{p'}$ is the double cover (where points in O are identified appropriately) of the fiber $\hat{\xi}_p$. Thus we obtain

$$\sum_{c\in(\hat{\theta}_{1,p}\times I)\cap\hat{\theta}_{2,p}} n(c) = \frac{(-1)^{g-1}}{2}\langle(\hat{\theta}_{1,p'}\times I),\hat{\theta}_{2,p'}\rangle_{\hat{\xi}_{p'}}.$$

(Note that if $\hat{\xi}_p \cap O$ is not empty, the error in the count of lifts of points $q \in (\hat{\theta}_1 \times I)\cap\hat{\theta}_2 \cap O$ is offset by the fraction in the definition of $s(q)$.)

Let $\beta\colon I \to \mathscr{L}_{p'}^{\mathbb{C}}$ be a path of complex lagrangians from $\eta_{1,p'}^{\dagger}$ to $\eta_{2,p'}$ transverse to $\eta_{2,p'}$. (This is possible since the space of complex lagrangians transverse to a fixed one is contractible.) Let δ be the loop $(\eta_{1,p'}\times I * \beta)$. Since $(\hat{\theta}_{1,p'}\times I) = \mathbb{C}P(\eta_{1,p'}\times I)$ and $\mathbb{C}P(\beta)$ is disjoint from $\hat{\theta}_{2,p'} = \mathbb{C}P(\eta_{2,p'})$, we have

$$\sum_{c\in(\hat{\theta}_{1,p}\times I)\cap\hat{\theta}_{2,p}} n(c) = \begin{cases} (-1)^{g-1}\langle\mathbb{C}P(\delta),\ \mathbb{C}P(\eta_{2,p'})\rangle_{\hat{\xi}_{p'}} & \text{if } p\notin P,\\ \dfrac{(-1)^{g-1}}{2}\langle\mathbb{C}P(\delta),\ \mathbb{C}P(\eta_{2,p'})\rangle_{\hat{\xi}_{p'}} & \text{if } p\in P.\end{cases}$$

We now consider $I(p)-I^{\dagger}(p)$. Since special isotopies are fixed on S, we can take the same curves and surfaces for computing $I(p)$ and $I^{\dagger}(p)$. It follows that

$$I(p)-I^{\dagger}(p) = \begin{cases} -\frac{1}{2}(\Phi-\Phi^{\dagger}) & \text{if } p\notin P,\\ -\frac{1}{4}(\Phi-\Phi^{\dagger}) & \text{if } p\in P,\end{cases}$$

where Φ is computed using $\eta_1\times\{0\}$, while $\Phi^{\dagger}$ is computed using $\eta_1\times\{1\}$.

Note that $\Phi-\Phi^{\dagger} = (\Phi_{|\gamma'}-\Phi^{\dagger}_{|\gamma'}) + (\Phi_{|\gamma''}-\Phi^{\dagger}_{|\gamma''})$. We look first at the trivializations over γ'.

Choose any trivialization $\psi\colon \det^1(\nu_{SO(3)})_{|\gamma'} \to S^1$. Then the difference $\Phi_{|\gamma'} - \Phi^{\dagger}_{|\gamma'}$ is the difference of the degrees $\deg\psi(x)-\deg\psi(x^{\dagger})$ where

$$\begin{aligned} x = {} & (-\det{}^1(\eta_2)_{|\alpha_2'}) * (-\det{}^1(P(\eta_{1,1},\eta_{2,1})))\\ & * (\det{}^1(\eta_1)_{|\alpha_1'}) * (\det{}^1(P(\eta_{1,p'},\eta_{2,p'})))\end{aligned}$$

and $x^{\dagger}$ is the corresponding path using $\eta_1^{\dagger}$ in place of η_1. The one-parameter family of paths

$$\mathscr{F}_t = (-\det{}^1(\eta_2)_{|\alpha_2'}) * (-\det{}^1(P(\eta_{1,1}\times\{t\},\eta_{2,1}))) * (\det{}^1(\eta_1\times\{t\})_{|\alpha_1'})$$

allows us to reduce to $\Phi_{|_{y'}} - \Phi^{\dagger}_{|_{y'}} = \deg \psi(y)$ where

$$y = (-\det{}^1(\eta_{1,p'} \times I)) * \det{}^1(P(\eta_{1,p'}, \eta_{2,p'})) * (-\det{}^1(P(\eta^{\dagger}_{1,p'}, \eta_{2,p'}))).$$

Recall that β is a path of complex lagrangians transverse to $\eta_{2,p'}$ from $\eta^{\dagger}_{1,p'}$ to $\eta_{1,p'}$. The family of paths $P(\beta_t, \eta_{2,p'})$ yields a homotopy relative the endpoints from $(\det^1(P(\eta_{1,p'}, \eta_{2,p'}))) * (-\det^1(P(\eta^{\dagger}_{1,p'}, \eta_{2,p'})))$ to $-\det^1(\beta)$. It follows that

$$\Phi_{|_{y'}} - \Phi^{\dagger}_{|_{y'}} = \deg(-\det{}^1(\eta_{1,p'} \times I) * -\det{}^1\beta) = \deg(\det{}^1(-\delta)).$$

By equivariance, $\Phi_{|_{y''}} - \Phi^{\dagger}_{|_{y''}} = \Phi_{|_{y'}} - \Phi^{\dagger}_{|_{y'}}$, so

$$I(p) - I^{\dagger}(p) = \begin{cases} \deg(\det{}^1(\delta)) & \text{if } p \notin P, \\ \frac{1}{2}\deg(\det{}^1(\delta)), & \text{if } p \in P. \end{cases}$$

By Lemma 1.11 of [W], $\deg(\det^1(\delta)) = (-1)^{g-1}\langle \mathbb{C}P(\delta), \mathbb{C}P(\eta_{2,p'})\rangle_{\hat{\xi}_{p'}}$. Thus, $\lambda_{SO(3)}(M)$ does not depend on the choice of special isotopy.

3. Calculation of the SO(3)-invariant

In this section, we view $\lambda_{SO(3)}(M)$ as a sum of family of invariants

$$\{\lambda_\alpha(M)\}_{\alpha \in H^2(M;\mathbb{Z}/2)}.$$

In 3.1, we define $\lambda_\alpha(M)$ and give an alternative description of $\lambda_\alpha(M)$ in terms of SU(2)-theory. Section 3.2 is devoted to proving

$$\lambda_\alpha(M) = \frac{1}{|H^1(M;\mathbb{Z}/2)|}\lambda_{SU(2)}(M).$$

3.1. **Lifting to** SU(2)**-theory.** For $\alpha \in H^2(M;\mathbb{Z}/2)$, let

$$\lambda_\alpha(M) = \frac{1}{|H_1(M;\mathbb{Z}/2)|}\left(\sum_{p_\alpha \in Q^-_{1,SO(3)} \cap Q^-_{2,SO(3)}} \operatorname{sign}(p_\alpha) + \sum_{p_\alpha \in (OQ_{1,SO(3)} \cap OQ_{2,SO(3)}) - S_{SO(3)}} s(p_\alpha) + \sum_{p_\alpha \in T_{1,SO(3)} \cap T_{2,SO(3)}} I(p_\alpha)\right)$$

where the sums on the right-hand side are taken over p_α with $\omega_2(p_\alpha) = \alpha$. This is well-defined given a choice of special isotopy, and

$$\sum_{\alpha \in H^2(M;\mathbb{Z}/2)} \lambda_\alpha(M) = \lambda_{SO(3)}(M).$$

We wish to represent $\lambda_\alpha(M)$ in terms of SU(2)-theory. This will allow us to show that the $\lambda_\alpha(M)$ are in fact well defined invariants of M and to compute the invariants. Section 2.1 provides the setting for lifting these invariants to SU(2). However there is still some work to be done.

Recall that the orientation $[R_G]$ is determined by the sign of ω_G, which is given by the identification of $H^2(\pi_1(\Sigma), \mathbb{R})$ with $\mathbb{R}$. This identification depends only on the orientation of Σ and M; hence the map $R_{SU(2)} \to$

$R_{SO(3)}$ is orientation preserving. Then the map $\iota_{\beta_\alpha} Q_{2,SU(2)} \to Q_{2,SO(3)}$ preserves (respectively, reverses) orientation if $[\iota_{\beta_\alpha} Q_{2,SU(2)}] = [Q_{2,SU(2)}]$ (respectively, $[\iota_{\beta_\alpha} Q_{2,SU(2)}] = -[Q_{2,SU(2)}]$). In fact, $[\iota_{\beta_\alpha} Q_{2,SU(2)}] = [Q_{2,SU(2)}]$ for all $\beta_\alpha \in H^1(\Sigma; \mathbb{Z}/2)$. To see this, consider the map $\iota_{\beta_\alpha}: Q^\#_{2,SU(2)} \to R^\#_{SU(2)}$. We have $Q^\#_{2,SU(2)} = (SU(2))^g$, and $\iota_{\beta_\alpha}(X_1, \dots, X_g) = (\beta_\alpha(x_1)X_1, \dots, \beta_\alpha(x_g)X_g)$, where x_i is a generator of $\pi_1(H_1)$. Thus, ι_{β_α} acts on each factor of $(SU(2))^g$ as either the identity or the antipodal map, both of which are orientation-preserving. It follows that $\iota_{\beta_\alpha}: Q^\#_{2,SU(2)} \to R^\#_{SU(2)}$ and hence $\iota_{\beta_\alpha}: Q_{2,SU(2)} \to R_{SU(2)}$ are orientation-preserving.

Note that a special isotopy $h: R_{SO(3)} \times I \to R_{SO(3)}$ induces a special isotopy $\bar{h}: R_{SU(2)} \times I \to R_{SU(2)}$ which is equivariant with respect to the action of $H^1(\Sigma; \mathbb{Z}/2)$. If $h(Q_{1,SO(3)})$ is in general position with $Q_{2,SO(3)}$, then $\bar{h}(Q_{1,SU(2)})$ is in general position with $\iota_{\beta_\alpha} Q_{2,SU(2)}$ for all $\beta_\alpha \in H^1(\Sigma; \mathbb{Z}/2)$. Assume henceforth that $Q_{1,SO(3)}$ has been isotoped so that it is in general position with $Q_{2,SO(3)}$ and that $Q_{1,SU(2)}$ has been isotoped via the induced special isotopy.

Let $\bar{p} \in Q^-_{1,SU(2)} \cap \iota_{\beta_\alpha} Q^-_{2,SU(2)}$ and let p be its image in $Q_{1,SO(3)} \cap Q_{2,SO(3)}$. By the previous remarks on orientations and by the definition of $s(p)$ for $p \in (OQ_{1,SO(3)} \cap OQ_{2,SO(3)}) - S_{SO(3)}$, we have

$$\operatorname{sign}(\bar{p}) = \begin{cases} \operatorname{sign}(p) & \text{if } p \in Q^-_{1,SO(3)} \cap Q^-_{2,SO(3)}, \\ 2s(p) & \text{if } p \in OQ^-_{1,SO(3)} \cap OQ^-_{2,SO(3)}, \\ 4s(p) & \text{if } p \in AQ^-_{1,SO(3)} \cap AQ^-_{2,SO(3)}. \end{cases}$$

It follows that

$$\frac{1}{|H^1(M;\mathbb{Z}/2)|} \sum_{\bar{p} \in Q^-_{1,SU(2)} \cap \iota_{\beta_\alpha} Q^-_{2,SU(2)}} \operatorname{sign}(\bar{p})$$
$$= \sum_{p_\alpha \in Q^-_{1,SO(3)} \cap Q^-_{2,SO(3)}} \operatorname{sign}(p_\alpha) + \sum_{p_\alpha \in (OQ_{1,SO(3)} \cap OQ_{2,SO(3)}) - S_{SO(3)}} s(p_\alpha).$$

We wish to define numbers $I(\bar{p})$ for $\bar{p} \in Q^-_{1,SU(2)} \cap \iota_{\beta_\alpha} Q^-_{2,SU(2)}$ and show that for any choice of β_α,

$$\frac{1}{|H^1(M;\mathbb{Z}/2)|} \sum_{\bar{p} \in T_{1,SU(2)} \cap \iota_{\beta_\alpha} T_{2,SU(2)}} I(\bar{p}) = \sum_{p_\alpha \in T_{1,SO(3)} \cap T_{2,SO(3)}} I(p_\alpha).$$

Note that the trivial representation id is not in $\iota_{\beta_\alpha} T_{2,SU(2)}$ for $\alpha \neq 0$, so we cannot define $I(\bar{p})$ in quite the same way as before.

Let $p \in T_{1,SO(3)} \cap T_{2,SO(3)}$ with $\omega_2(p) = \alpha$. Let $\bar{p}$ be a lift in $T_{1,SU(2)} \cap \iota_{\beta_\alpha} T_{2,SU(2)}$. Note that $\alpha \neq 0$ implies $\bar{p} \notin Z_{SU(2)}$. Let $\bar{p}'$ be an inverse image of $\bar{p}$ in $\widetilde{S}_{SU(2)}$, and let p' be its image in $\widetilde{S}_{SO(3)}$. (So p' is an inverse image of p in $\widetilde{S}_{SO(3)}$.)

Let $\bar{\alpha}'_1$ be a path from id to $\bar{p}'$ in $\widetilde{T}_{2,SU(2)}$. Let $\bar{\alpha}'_2$ be a path from $\iota_{\beta_\alpha}(\text{id})$ to $\bar{p}'$ in $\iota_{\beta_\alpha} \widetilde{T}_{2,SU(2)}$. Let $\bar{\gamma}' = \bar{\alpha}'_1 * (-\bar{\alpha}'_2)$. Then the image γ' of $\bar{\gamma}'$ in $\widetilde{S}_{SO(3)}$ is a closed path from id to p' to id as in the definition of $I(p)$ for $p \in T_{1,SO(3)} \cap T_{2,SO(3)}$. $\bar{\gamma}'$ is the unique lift of γ' to a path beginning at id.

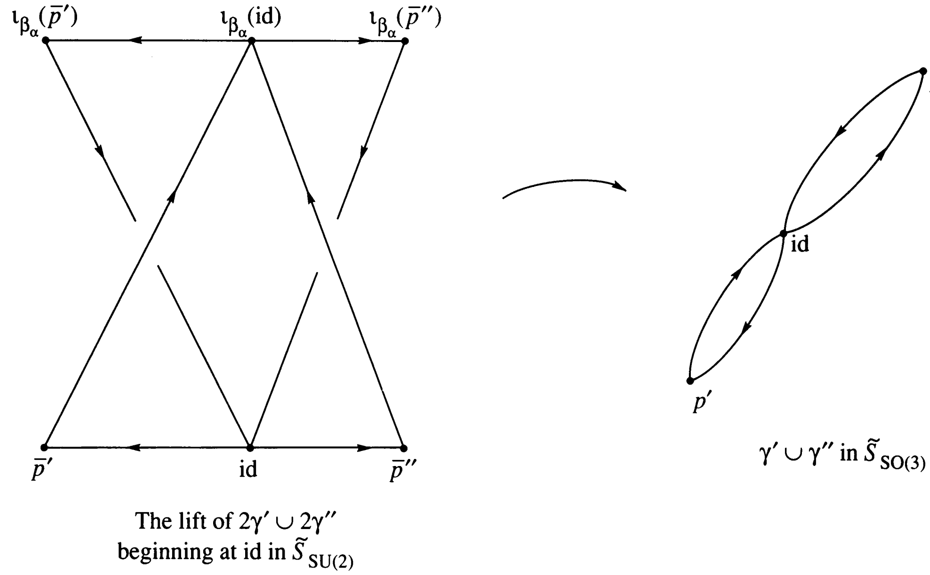


FIGURE 3.1.1

We claim that while γ' does not lift to a closed path in $\widetilde{S}_{\mathrm{SU}(2)}$, $2\gamma'$ does. To see this, consider the inverse images of γ' in $\widetilde{S}_{\mathrm{SU}(2)}$. There are 2^{2g} of them, one beginning at each inverse image $\iota_\beta(\text{id})$ of id, where β is any element of $H^1(\Sigma; \mathbb{Z}/2)$. The inverse image beginning at $\iota_\beta(\text{id})$ is a path from $\iota_\beta(\text{id})$ to $\iota_\beta(\bar{p}')$ in $\iota_\beta \widetilde{T}_{1,\mathrm{SU}(2)}$ followed by a path from $\iota_\beta(\bar{p}')$ to $\iota_\beta(\iota_{\beta_\alpha}\text{id})$ in $\iota_\beta \iota_{\beta_\alpha} \widetilde{T}_{2,\mathrm{SU}(2)}$. In particular, the inverse image beginning at $\iota_{\beta_\alpha}(\text{id})$ ends at $\iota_{\beta_\alpha}(\iota_{\beta_\alpha}\text{id}) = \text{id}$. Thus, the inverse image of $2\gamma'$ which begins at id also ends at id. Call this path $\overline{2\gamma'}$. Let $\overline{2\gamma''} = \tau_{\mathrm{SU}(2)}(\overline{2\gamma'})$. Choose a surface E in $\widetilde{S}_{\mathrm{SU}(2)}$ with $\partial E = \overline{2\gamma'} \cup \overline{2\gamma''}$. (See Figure 3.1.1.)

Define trivializations $\Phi_\pm$ of $\det^1(\nu_{\mathrm{SU}(2)})_{|\bar{\gamma}' \cup \bar{\gamma}''}$ as for $I(p)$, where $p \in T_{1,\mathrm{SO}(3)} \cap T_{2,\mathrm{SO}(3)}$, using the usual patching procedure at id, $\bar{p}'$, $\iota_{\beta_\alpha}(\text{id})$, $\iota_{\beta_\alpha}(\bar{p}')$, $\bar{p}''$, and $\iota_{\beta_\alpha}(\bar{p}'')$ and using the sections $\det^1(\iota_{\beta_\alpha}\eta_{j,\mathrm{SU}(2)})$ over $\iota_{\beta_\alpha}\widetilde{T}_{j,\mathrm{SU}(2)}$. (Note that $\iota_{\beta_\alpha}\eta_{j,\mathrm{SU}(2)}$ is a complex lagrangian subspace of $\nu_{\mathrm{SU}(2)}$, so this makes sense.)

Finally, define

$$I(\bar{p}) = \frac{1}{2}\left(\frac{1}{2}\left(c_1(\det{}^1(\nu_{\mathrm{SU}(2)})_{|E}, \Phi) - \int_E \omega_{\mathrm{SU}(2)}\right)\right)$$
$$= \frac{1}{4}\left(c_1(\det{}^1(\nu_{\mathrm{SU}(2)})_{|E}, \Phi) - \int_E \omega_{\mathrm{SU}(2)}\right).$$

The proof that these numbers are well defined is similar to that for $I(p)$ where $p \in T_{1,\mathrm{SO}(3)} \cap T_{2,\mathrm{SO}(3)}$. We show

Lemma 3.1. *Let $p \in S_{\mathrm{SO}(3)}$, and let $\bar{p}$ be a lift to $T_{1,\mathrm{SU}(2)} \cap \iota_{\beta_\alpha} T_{2,\mathrm{SU}(2)}$, (where $\beta_\alpha = 0$ if $\alpha = 0$). Then*

$$I(p) = \begin{cases} I(\bar{p}) & \text{if } p \notin P^-_{\mathrm{SO}(3)}, \\ \frac{1}{2} I(\bar{p}) & \text{if } p \in P^-_{\mathrm{SO}(3)}. \end{cases}$$

Proof. If $\bar{p} = \text{id}_{\mathrm{SU}(2)}$, this is trivial. Otherwise:

Let $\overline{p}'$ and $\overline{p}''$ be the lifts of $\overline{p}$ to $\widetilde{S}_{\mathrm{SU}(2)}$, and let p' and p'' be their respective images in $\widetilde{S}_{\mathrm{SO}(3)}$. (Then p' and p'' are the lifts of p to $\widetilde{S}_{\mathrm{SO}(3)}$, since clearly p' and p'' map to p in $S_{\mathrm{SO}(3)}$ and $p' = p''$ if and only if $p \in P^{-}_{\mathrm{SO}(3)}$.)

Fix arcs $\overline{\alpha}_j$ and a surface $\overline{E}$ for calculating $I(\overline{p})$. Let α_j be the images of $\overline{\alpha}_j$ and let E be the image of $\overline{E}$ in $\widetilde{S}_{\mathrm{SO}(3)}$. (If the image of $\overline{E}$ winds around a torus in $\widetilde{S}_{\mathrm{SO}(3)}$, we view E as a surface with the appropriate boundary and the appropriate number of copies of the torus attached.) Then these are suitable for calculating $I(p)$ if $p \neq \mathrm{id}_{\mathrm{SO}(3)}$. Let $\overline{\Phi}$ be the trivialization of $\partial\overline{E}$ used for calculating $I(\overline{p})$, and let Φ be the trivialization of ∂E used for calculating $I(p)$.

We show

$$c_1(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}}), \overline{\Phi}) = \begin{cases} c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) & \text{if } \alpha = 0, \\ 2c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) & \text{if } \alpha \neq 0, \end{cases}$$

and

$$\int_{\overline{E}} \omega_{\mathrm{SU}(2)} = \begin{cases} \displaystyle\int_E \omega_{\mathrm{SO}(3)} & \text{if } \alpha = 0, \\ \displaystyle 2\int_E \omega_{\mathrm{SO}(3)} & \text{if } \alpha \neq 0. \end{cases}$$

If $p = \mathrm{id}_{\mathrm{SO}(3)}$, we show further that

$$c_1(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}}), \overline{\Phi}) - \int_{\overline{E}} \omega_{\mathrm{SU}(2)} = 0.$$

This will prove the lemma.

The map $\overline{E} \to E$ is a homeomorphism if $\alpha = 0$; otherwise it is a double cover. (Again, if necessary we view E as having a closed surface attached.) To see this, recall that covering space theory implies that there are 2^{2g} lifts of E to $\widetilde{S}_{\mathrm{SU}(2)}$ with boundaries $\overline{\gamma}_{i,1} \cup \overline{\gamma}_{i,2}$ which are lifts of $\gamma_1 \cup \gamma_2$ based at the 2^{2g} distinct lifts of $\mathrm{id}_{\mathrm{SO}(3)}$. If $\alpha = 0$, $\overline{E}$ is the unique lift of E with boundary $\overline{\gamma}_1 \cup \overline{\gamma}_2$, where $\overline{\gamma}_i$ is a loop based at the trivial representation $\mathrm{id}_{\mathrm{SU}(2)}$. If $\alpha \neq 0$, $\overline{E}$ is the unique lift of E with boundary $\overline{\gamma}_1 \cup \overline{\gamma}_2 \cup \iota_{\beta_\alpha}\overline{\gamma}_1 \cup \iota_{\beta_\alpha}\overline{\gamma}_2$, where $\overline{\gamma}_i$ (respectively, $\iota_{\beta_\alpha}\overline{\gamma}_i$) is a path based at $\mathrm{id}_{\mathrm{SU}(2)}$ (respectively, $\iota_{\beta_\alpha}\mathrm{id}_{\mathrm{SU}(2)}$). It follows that

$$c_1(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}})) = \begin{cases} c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E})) & \text{if } \alpha = 0, \\ 2c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E})) & \text{if } \alpha \neq 0. \end{cases}$$

By Lemma 2.5, $\det{}^1(\eta_{j,\mathrm{SU}(2)|_{\overline{\alpha}_j}})$ maps to $\det{}^1(\eta_{j,\mathrm{SO}(3)|_{\alpha_j}})$ if $\alpha = 0$. If $\alpha \neq 0$, we know ι_{β_α} takes $Q_{j,\mathrm{SU}(2)}$ to $\iota_{\beta_\alpha}Q_{j,\mathrm{SU}(2)}$ preserving orientations, so it takes the Zariski normal bundle of $T_{j,\mathrm{SU}(2)}$ in $Q_{j,\mathrm{SU}(2)}$ to the Zariski normal bundle of $\iota_{\beta_\alpha}T_{j,\mathrm{SU}(2)}$ in $\iota_{\beta_\alpha}Q_{j,\mathrm{SU}(2)}$. This shows that $\det^1(\iota_{\beta_\alpha}\eta_{j,\mathrm{SU}(2)|_{\overline{\alpha}_j}})$ maps to $\det{}^1(\eta_{j,\mathrm{SO}(3)|_{\alpha_j}})$ away from points in $\widetilde{Z}_{\mathrm{SU}(2)}$. Lemma 2.5 insures that this works over points in $\widetilde{Z}_{\mathrm{SU}(2)}$ as well.

We claim that $\overline{\Phi}$ maps to Φ if $\alpha = 0$ and 2Φ if $\alpha \neq 0$. We must show that the patching procedure used at $\mathrm{id}_{\mathrm{SU}(2)}$, $\overline{p}'$, $\overline{p}''$, and, if relevant, $\iota_{\beta_\alpha}\overline{p}'$ and $\iota_{\beta_\alpha}\overline{p}''$ agrees with that used at $\mathrm{id}_{\mathrm{SO}(3)}$, p', and p''. To show this, it is sufficient to show that the fiber at a point $\overline{q} \in \widetilde{S}_{\mathrm{SU}(2)}$ is mapped homeomorphically onto the fiber at its image $q \in \widetilde{S}_{\mathrm{SO}(3)}$.

This is clear, since a point x in the fiber at q can have at most 2^{2g} inverse images in $\widetilde{S}_{\mathrm{SU}(2)}$, yet it must have an inverse image in the fiber over each of the 2^{2g} inverse images of q. It follows that $\overline{\Phi}$ maps to Φ (respectively, 2Φ) if $\alpha = 0$ (respectively, $\alpha \neq 0$), and hence

$$c_1(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}}), \overline{\Phi}) = \begin{cases} c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) & \text{if } \alpha = 0, \\ 2c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) & \text{if } \alpha \neq 0. \end{cases}$$

For the integrals, we have

$$\begin{aligned} \int_{\overline{E}} \omega_{\mathrm{SU}(2)} &= \deg(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}}) \to \det{}^1(\nu_{\mathrm{SO}(3)|_E})) \int_E \omega_{\mathrm{SO}(3)} \\ &= \begin{cases} \displaystyle\int_E \omega_{\mathrm{SO}(3)} & \text{if } \alpha = 0, \\ \displaystyle 2\int_E \omega_{\mathrm{SO}(3)} & \text{if } \alpha \neq 0. \end{cases} \end{aligned}$$

Finally, if $p = \mathrm{id}_{\mathrm{SO}(3)}$ (and hence $\alpha = 0$), note that if E and Φ are chosen in the usual way, the difference $c_1(\det^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) - \int_E \omega_{\mathrm{SO}(3)}$ is independent of the choice of arcs α_j from $\mathrm{id}_{\mathrm{SO}(3)}$ to $\mathrm{id}_{\mathrm{SO}(3)}$ by the usual argument. In particular, we may take α_j and E to be degenerate, consisting of the single point $\mathrm{id}_{\mathrm{SO}(3)}$. It follows that

$$\begin{aligned} c_1(\det{}^1(\nu_{\mathrm{SU}(2)|_{\overline{E}}}), \overline{\Phi}) - \int_{\overline{E}} \omega_{\mathrm{SU}(2)} &= c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_E}), \Phi) - \int_E \omega_{\mathrm{SO}(3)} \\ &= c_1(\det{}^1(\nu_{\mathrm{SO}(3)|_{\mathrm{point}}}), \mathrm{point}) - \int_{\mathrm{point}} \omega_{\mathrm{SO}(3)} = 0. \quad \square \end{aligned}$$

Corollary 3.2. *For any $\mathbb{Q}HS\ M$,*

$$\lambda_0(M) = \frac{1}{|H^1(M, \mathbb{Z}/2)|}\lambda_{\mathrm{SU}(2)}(M).$$

Finally, this representation of $\lambda_\alpha(M)$ enables us to show

Theorem 3.3. *$\lambda_\alpha(M)$ is a well-defined invariant of M.*

This can be shown using arguments similar to those in §2. The proof of special isotopy invariance is carried out in $\widetilde{S}_{\mathrm{SU}(2)}$ rather than in $\widetilde{S}_{\mathrm{SO}(3)}$. (We obtain copies of the loop δ at both $\overline{p}'$ and $\iota_{\beta_\alpha}\overline{p}'$; carry out the same argument for both loops.) In particular, it is clear that $\lambda_\alpha(M)$ is independent of the choice of special isotopy of $R_{\mathrm{SU}(2)}$ used to put $Q_{1,\mathrm{SU}(2)}$ into general position with $\iota_{\beta_\alpha}Q_{2,\mathrm{SU}(2)}$; this special isotopy need not be equivariant with respect to the $H^1(M; \mathbb{Z}/2)$-action.

We are now ready to calculate these invariants.

3.2. Calculation of $\lambda_{\mathrm{SO}(3)}(M)$.

In this section we show

Theorem 3.4. *Let M be a $\mathbb{Q}HS$. Then $\lambda_{\mathrm{SO}(3)}(M) = \lambda_{\mathrm{SU}(2)}(M)$.*

Throughout this section, representation spaces are assumed to consist of SU(2)-representations unless otherwise stated; we omit the subscripts.

We adopt the following notation:

$$\langle Q_1, \iota_{\beta_\alpha} Q_2\rangle = \sum_{p\in Q_1^-\cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \sum_{p\in T_1\cap\iota_{\beta_\alpha} T_2} I(p).$$

Let $K \subset M$ be a knot in a $\mathbb{Q}HS$, and let $N = M - \operatorname{nbhd}(K)$. Let $l \in H_1(\partial N; \mathbb{Z})$ be a longitude of K, and let a and b be primitive homology classes in $H_1(\partial N; \mathbb{Z})$ such that $\langle a, l\rangle\langle b, l\rangle > 0$ and $\langle a, b\rangle = 1$. For primitive $x \in H_1(\partial N; \mathbb{Z})$, let N_x denote the manifold obtained by Dehn surgery along x.

Let U be a tubular neighborhood of K. We choose a Heegaard decomposition so that U corresponds to a handle of the first handlebody. More precisely, let γ_a and γ_b be simple closed curves on ∂U representing a and b respectively and intersecting once transversely. Attach 1-handles to U disjoint from γ_a and γ_b so that $H_1 = U \cup \{\text{1-handles}\}$ and $H_2 = \overline{M - H_1}$ form a Heegaard splitting of M.

Let $\gamma \subset \Sigma$ be the boundary of a regular neighborhood of $\gamma_a \cup \gamma_b$ in Σ. Let $\delta_1, \delta_2, \dots, \delta_{g-1} \subset \Sigma$ be boundaries of discs transverse to the cores of the 1-handles.

Let $z \in H_1(\partial U; \mathbb{Z})$ be primitive, and let $\gamma_z \subset \partial U$ be a simple closed curve representing z and disjoint from γ. Let

$$Q_1^z = \{[\rho] \in R | \rho(\gamma_z) = 1 \text{ and } \rho(\delta_i) = 1 \text{ for } 1 \le i \le g-1\},$$
$$Q_1^{-z} = \{[\rho] \in R | \rho(\gamma_z) = -1 \text{ and } \rho(\delta_i) = 1 \text{ for } 1 \le i \le g-1\}.$$

(Of course, $Q_1^{-z} = \iota_x Q_1^z$ for appropriate $x \in H^2(\Sigma; \mathbb{Z}/2)$. However, we will treat it differently in what follows, so we adopt this new notation.) Note that Q_1^z and Q_2 are representation spaces corresponding to a Heegaard decomposition of the manifold which is the result of z-surgery along K. (That is, $\pm N_z$.) It follows that for $\alpha \in H^2(N_z; \mathbb{Z}/2)$ we have

$$\lambda_\alpha(N_z) = \pm\frac{\sum_{p\in Q_1^{z,-}\cap\iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \sum_{p\in T_1^z\cap\iota_{\beta_\alpha} T_2} I(p)}{|H^1(N_z; \mathbb{Z}/2)|\,|H_1(N_z; \mathbb{Z})|}.$$

Let H denote the cohomology group from the following list which has the largest order: $H^2(N_a; \mathbb{Z}/2)$, $H^2(N_b; \mathbb{Z}/2)$, or $H^2(N_{a+b}; \mathbb{Z}/2)$. Note that for any primitive $z \in H_1(\partial U; \mathbb{Z})$, $H^2(N_z; \mathbb{Z}/2)$ as well as $H^2(M; \mathbb{Z}/2)$ either is naturally isomorphic to or includes naturally into H.

Assume $z \in H_1(\partial U; \mathbb{Z})$ is primitive and satisfies $H \cong H^2(N_z; \mathbb{Z}/2) \times \mathbb{Z}/2$. If $\alpha = (\alpha', 1) \in H^2(N_z; \mathbb{Z}/2) \times \mathbb{Z}/2$, then any inverse image β_α of α is an inverse image of α' under the map

$$H^1(\Sigma; \mathbb{Z}/2) \to H^2(N_z; \mathbb{Z}/2) \times \mathbb{Z}/2 \to H^2(N_z; \mathbb{Z}/2)$$

where the second homomorphism is the projection onto the first factor and the composite is the boundary operator in the Mayer-Vietoris cohomology sequence for the triple $(N_z; H_1, H_2)$. It follows that

$$|\lambda_{\alpha'}(N_z)| = \frac{|\langle Q_1^z, \iota_{\beta_\alpha} Q_2\rangle|}{|H^1(N_z; \mathbb{Z}/2)|\,|H_1(N_z; \mathbb{Z})|}.$$

In what follows, we abuse notation and write $\lambda_\alpha(N_z)$ for any $\alpha \in H$. If $\alpha \notin H^2(N_z; \mathbb{Z}/2)$, we understand this to mean $\lambda_{\alpha'}(N_z)$, where $\alpha = (\alpha', 1)$. Also, we write $\overline{\lambda}_\alpha(X)$ for $|H^1(X; \mathbb{Z}/2)|\,|H_1(X; \mathbb{Z})|\lambda_\alpha(X)$. We show

Lemma 3.5 (due to Walker for $\alpha = 0$). *For* $\alpha \in H$,

$$\overline{\lambda}_\alpha(N_{a+b}) = \overline{\lambda}_\alpha(N_a) + \overline{\lambda}_\alpha(N_b) + \tfrac{1}{6}(|H_1(N_a;\mathbb{Z})| - |H_1(N_b;\mathbb{Z})|).$$

Proof. Assume that $\iota_{\beta_\alpha} Q_2$ has been isotoped equivariantly with respect to the $H^1(M;\mathbb{Z}/2)$ action into general position with $Q_1^{\pm z}$ for all primitive z in $H_1(\partial U;\mathbb{Z})$ with $\langle z, l\rangle \neq 0$ and for all α.

Let h be a right-handed Dehn twist along γ_b. Further, let h denote the induced maps on homology groups, homotopy groups, and representation spaces. Note that $h(a+b) = a$, so $h(Q_1^{a+b}) = Q_1^a$. Thus, we wish to show

$$\langle Q_1^{a+b}, \iota_{\beta_\alpha} Q_2\rangle - \langle h(Q_1^{a+b}), \iota_{\beta_\alpha} Q_2\rangle = \langle Q_1^b, \iota_{\beta_\alpha} Q_2\rangle + \tfrac{1}{6}(|H_1(N_a;\mathbb{Z})| - |H_1(N_b;\mathbb{Z})|).$$

To measure the difference $\langle Q_1^{a+b}, \iota_{\beta_\alpha} Q_2\rangle - \langle h(Q_1^{a+b}), \iota_{\beta_\alpha} Q_2\rangle$, we use the isotopy $\{h_t\}_{t\in[0,1)}$ of R induced by $h_t^{\#}\colon R^{\#} \to R^{\#}$, where

$$h_t^{\#}(\rho(x)) = \rho(x) \quad \text{if } x \in \pi_1(\Sigma_1),$$
$$h_t^{\#}(\rho(\gamma_a)) = \rho(\gamma_b)^{f(\rho(\gamma_b),t)^t}\rho(\gamma_a).$$

Here, Σ_1 denotes Σ cut apart along γ_b and $f\colon \mathrm{SU}(2)\times[0,1) \to [0,1]$ is a smooth, Ad-invariant function such that $f(X,t) = 1$ if $\operatorname{trace}(X) \geq -1-t$, $f(-I,t) = 0$ for all t, and $f(\cdot,t)$ is monotonic for all t. An adaptation of a proof of Walker for the $\alpha = 0$ case [W, Lemma 3.36] shows that this is a "not-so-special" isotopy of R; that is, $h_t(Q_1^{a+b})$ is transverse to $\iota_{\beta_\alpha} Q_2$ at $Q_1^{a+b} \cap \iota_{\beta_\alpha} Q_2 \cap Z$ for all t, $h_t(T_1^{a+b})$ is transverse to $\iota_{\beta_\alpha} T_2$ in S for all t, and h_t induces a symplectic bundle map of ν for all t.

As in the case $\alpha = 0$, an argument similar to that used to prove special isotopy invariance shows that

$$\sum_{p\in Q_1^- \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \begin{cases} \displaystyle\sum_{p\in T_1\cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu_{|_E}), \Phi) & \text{if } \alpha = 0 \\ \displaystyle\sum_{p\in T_1\cap T_2} \frac{1}{2} c_1(\det{}^1(\nu_{|_E}), \Phi) & \text{if } \alpha = 0 \end{cases}$$

is invariant under not-so-special isotopies as long as we use the induced arcs and surfaces to calculate the c_1-terms. In other words, we know that

$$\sum_{p\in h_t(Q_1^{a+b,-}) \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \begin{cases} \displaystyle\sum_{p\in h_t(T_1)\cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu_{|_{h_t(E)}}), h_t(\Phi)) & \text{if } a \neq 0 \\ \displaystyle\sum_{p\in h_t(T_1)\cap T_2} \frac{1}{2} c_1(\det{}^1(\nu_{|_{h_t(E)}}), h_t(\Phi)) & \text{if } \alpha = 0 \end{cases}$$
$$= \sum_{p\in Q_1^{a+b,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \begin{cases} \displaystyle\sum_{p\in T_1\cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu_{|_E}), \Phi) & \text{if } \alpha \neq 0 \\ \displaystyle\sum_{p\in T_1\cap T_2} \frac{1}{2} c_1(\det{}^1(\nu_{|_E}), \Phi) & \text{if } \alpha = 0 \end{cases}$$

for $0 \leq t < 1$, where E and Φ are the surfaces and trivializations used for calculating the c_1-terms for $p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2$ and $h_t(E)$ and $h_t(\Phi)$ are the images of E and Φ under h_t.

Thus, we must first compare

$$\lim_{t\to 1} \sum_{p\in h_t(Q_1^{a+b,-})\cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \lim_{t\to 1} \begin{cases} \displaystyle\sum_{p\in h_t(T_1)\cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu_{|h_t(E)}), h_t(\Phi) & \\ \qquad\qquad\qquad \text{if } \alpha \neq 0, & \\ \displaystyle\sum_{p\in h_t(T_1)\cap T_2} \frac{1}{2} c_1(\det{}^1(\nu_{|h_t(E)}), h_t(\Phi)) & \\ \qquad\qquad\qquad \text{if } \alpha = 0, & \end{cases}$$

with the corresponding sums for points in $Q_1^a \cap \iota_{\beta_\alpha} Q_2$ and $Q_1^b \cap \iota_{\beta_\alpha} Q_2$. We then compare the corresponding integral terms, thereby proving the lemma.

We begin with the $\operatorname{sign}(p)$- and c_1-terms. We remark that as t approaches 1, $h_t(Q_1^{a+b})$ approaches $\{[\rho] \in R | \rho(\gamma_a) = 1\} \cup \{[\rho] \in R | \rho(\gamma_b) = -1\} = Q_1^a \cup Q_1^{-b}$. Moreover, as $t \to 1$, $h_t \to h$ on $\{[\rho] \in R | \rho(\gamma_b) \neq -1\}$. Thus, intuitively, we should be comparing the $\operatorname{sign}(p)$- and c_1-terms of "$\langle Q_1^a, \iota_{\beta_\alpha} Q_2\rangle + \langle Q_1^{-b}, \iota_{\beta_\alpha} Q_2\rangle$" with those of $\langle Q_1^a, \iota_{\beta_\alpha} Q_2\rangle + \langle Q_1^b, \iota_{\beta_\alpha} Q_2\rangle$. Of course, we must make sense of the term $\langle Q_1^{-b}, \iota_{\beta_\alpha} Q_2\rangle$. Also, this will not quite work, since h_t acts on $\iota_{\beta_\alpha} Q_1^{a+b}$ differently than on Q_1^{a+b}, so the c_1-terms in the limit as $t \to 1$ may also involve terms from $\langle Q_1^{-a}, \iota_{\beta_\alpha} Q_2\rangle$ and $\langle Q_1^b, \iota_{\beta_\alpha} Q_2\rangle$. We start by determining the action of h_t on $\iota_{\beta_\alpha} Q_1^{a+b}$ as $t \to 1$.

First of all, it is clear from the above that $h(\iota_{\beta_\alpha} Q_1^{a+b})$ must be contained in the limit of $h_t(\iota_{\beta_\alpha} Q_1^{a+b})$. We have

$$h(\rho \otimes \beta_\alpha)(x) = (\rho \otimes \beta_\alpha)(x) \quad \text{if } x \in \pi_1(\Sigma_1),$$
$$h(\rho \otimes \beta_\alpha)(\gamma_a) = \begin{cases} \rho(\gamma_b\gamma_a) & \text{if } \beta_\alpha(\gamma_b\gamma_a) = 1, \\ -\rho(\gamma_b\gamma_a) & \text{if } \beta_\alpha(\gamma_b\gamma_a) = -1. \end{cases}$$

We see that

$$h(\iota_{\beta_\alpha} Q_1^{a+b}) = \begin{cases} \iota_y Q_1^a & \text{if } \beta_\alpha(a+b) = 1, \\ \iota_y Q_1^{-a} & \text{if } \beta_\alpha(a+b) = -1, \end{cases}$$

where y agrees with β_α except on loops representing nontrivial elements of $H_1(\partial U; \mathbb{Z})$ and y of such elements is 1.

In fact, the limit as $t \to 1$ of $h_t(\iota_{\beta_\alpha} Q_1^{a+b})$ is $\iota_y Q_1^{\varepsilon_1 a} \cup \iota_y Q_1^{-b}$ where ε_1 is determined by the previous paragraph. This can be shown exactly as in the SU(2) case. (See [W, proof of 3.36 and proof of 3.2].)

We now decide what we mean by $\langle Q_1^{-b}, \iota_{\beta_\alpha} Q_2\rangle$ and $\langle Q_1^{-a}, \iota_{\beta_\alpha} Q_2\rangle$. We define $I(p)$ for $p \in Q_1^{-b} \cap \iota_{\beta_\alpha} Q_2$. The definition of $I(p)$ for $p \in Q_1^{-a} \cap \iota_{\beta_\alpha} Q_2$ is analogous.

Note that $T_1^{-b} = \iota_x T_1^b$ for an appropriate $x \in H^1(\Sigma; \mathbb{Z}/2)$. However, we do *not* want to define $I(p)$ as in §3.1. (This is the reason for the change in notation!) Remember we are thinking of $Q_1^a \cup Q_1^{-b}$ as the limit as $t \to 1$ of $h_t(Q_1^{a+b})$; therefore our terms $I(p)$ should also be limits in some sense. We do the following:

Let $p \in T_1^{-b} \cap \iota_{\beta_\alpha} T_2$ and let $p', p'' \in \widetilde{T}_1^{-b} \cap \iota_{\beta_\alpha} \widetilde{T}_2$ be the lifts of p to $\widetilde{S}$, where $p' = p''$ if $p \in Z$. Choose an arc from id to the point x of intersection of $\widetilde{T}_1^a$ and $\widetilde{T}_1^{-b}$ in $\widetilde{T}_1^a$ and an arc from x to p' in $\widetilde{T}_1^{-b}$. Let γ_1 be the union of these, oriented from id to p'. (See Figure 3.2.1.) As usual, let γ_2 be an arc from p' to ι_{β_α} id in $\iota_{\beta_\alpha} T_2$. Let

$$\gamma' = \begin{cases} \gamma_1 * (-\gamma_2) & \text{if } \alpha = 0, \\ \gamma_1 * (-\gamma_2) * \iota_{\beta_\alpha}\gamma_1 * (-\iota_{\beta_\alpha}\gamma_2) & \text{otherwise.} \end{cases}$$

Let $\gamma'' = \tau(\gamma')$. Define the trivialization over γ' (respectively, γ'') by taking $\det^1(\eta_j)$ or $\det^1(\iota_{\beta_\alpha}\eta_j)$ over each of the arcs, patching in the usual way at p', id, $\iota_{\beta_\alpha}p'$, and ι_{β_α} id, and patching at x and $\iota_{\beta_\alpha}x$ so that the relative homotopy class of the resulting section agrees with the relative homotopy class of the limit as $t \to 1$ of the corresponding sections over the appropriate arcs in $h_t(T_1^{a+b}) \cup \iota_{\beta_\alpha} T_2$. Now define $I(p)$ in the usual manner, using these arcs and trivializations. We now know what we mean by $\langle Q_1^{-b}, \iota_{\beta_\alpha} Q_2\rangle$ and $\langle Q_1^{-a}, \iota_{\beta_\alpha} Q_2\rangle$. These numbers are independent of the choice of special isotopy used to put the representation spaces into general position by the usual argument.

Next we verify that

$$(1) \quad \sum_{p \in Q_1^{a+b,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \sum_{p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi)$$

$$(2) \quad = \begin{cases} \displaystyle\sum_{p \in Q_1^{a,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \sum_{p \in T_1^{a} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) \\ \displaystyle\quad + \sum_{p \in Q_1^{-b,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) \\ \displaystyle\quad + \sum_{p \in T_1^{-b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) \quad \text{if } \beta_\alpha(b) = 1. \\ \displaystyle\frac{1}{2} \sum_{p \in Q_1^{a,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) + \frac{1}{2} \sum_{p \in T_1^{a} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) \\ \displaystyle\quad + \frac{1}{2} \sum_{p \in Q_1^{-b,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) \\ \displaystyle\quad + \frac{1}{2} \sum_{p \in T_1^{-b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) + \frac{1}{2} \sum_{p \in Q_1^{-a,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) \\ \displaystyle\quad + \frac{1}{2} \sum_{p \in T_1^{-a} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) + \frac{1}{2} \sum_{p \in Q_1^{b,-} \cap \iota_{\beta_\alpha} Q_2^-} \operatorname{sign}(p) \\ \displaystyle\quad + \frac{1}{2} \sum_{p \in T_1^{b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} c_1(\det{}^1(\nu)_{|E_p}, \Phi) \quad \text{if } \beta_\alpha(b) = -1. \end{cases}$$

By general position, we know that for t sufficiently close to 1,

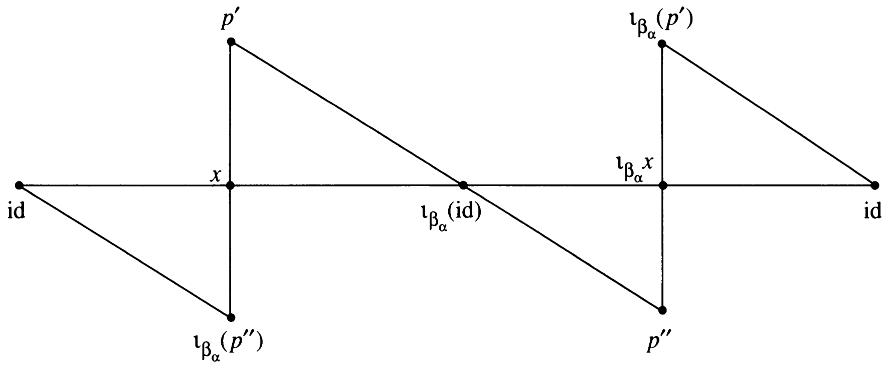


FIGURE 3.2.1

$$\sum_{p\in h_t(Q_1^{a+b,-})\cap \iota_{\beta_\alpha}Q_2^-} \operatorname{sign}(p) = \sum_{p\in Q_1^{a,-}\cap\iota_{\beta_\alpha}Q_2^-} \operatorname{sign}(p) + \sum_{p\in Q_1^{b,-}\cap\iota_{\beta_\alpha}Q_2} \operatorname{sign}(p).$$

Thus we must show

$$(3)\ \lim_{t\to 1} \sum_{p\in h_t(T_1^{a+b})\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi)$$

$$(4)\qquad = \begin{cases} \displaystyle\sum_{p\in T_1^a\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) \\ \displaystyle\quad + \sum_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) \quad \text{if } \beta_\alpha(b) = 1. \\ \displaystyle\frac{1}{2}\sum_{p\in T_1^a\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) + \frac{1}{2}\sum_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) \\ \displaystyle\quad + \frac{1}{2}\sum_{p\in T_1^{-a}\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) \\ \displaystyle\quad + \frac{1}{2}\sum_{p\in T_1^{b}\cap\iota_{\beta_\alpha}T_2} \frac{1}{4}c_1(\det{}^1(\nu)_{|E_p}, \Phi) \quad \text{if } \beta_\alpha(b) = -1. \end{cases}$$

To do so, we introduce terms $J(p', q')$ for $p' \in \widetilde{T}_1^x \cap \iota_{\beta_\alpha}\widetilde{T}_2$ and $q' \in \iota_{\beta_\alpha}\widetilde{T}_1^y \cap \widetilde{T}_2$, where $x = a$ or $-b$ and $y = b$ or $-a$.

Assume $\beta_\alpha \neq 0$. (If $\beta_\alpha = 0$, Walker has already calculated this difference in [W]. Also, the following works even if $\beta_\alpha = 0$ as long as we calculate $I(p)$ be doubling our arcs and surfaces and then dividing by 2.)

Let $p' \in \widetilde{T}_1^a \cap \iota_{\beta_\alpha}\widetilde{T}_2$ and let $q' \in \iota_{\beta_\alpha}\widetilde{T}_1^b \cap \widetilde{T}_2$. Choose an arc γ_1 from id to p' in $\widetilde{T}_1^a$ and an arc γ_2 from ι_{β_α}id to p' in $\iota_{\beta_\alpha}\widetilde{T}_2$. Let γ_3 be an arc from ι_{β_α}id to q' in $\iota_{\beta_\alpha}\widetilde{T}_1^b$ and γ_4 be an arc from id to q' in $\widetilde{T}_2$. Let $\gamma' = \gamma_1 * (-\gamma_2) * \gamma_3 * (-\gamma_4)$, and let $\gamma'' = \tau(\gamma')$, as shown in Figure 3.2.2. Let E be a surface in $\widetilde{S}$ with $\partial E = \gamma' \cup \gamma''$. Define a trivialization Φ over ∂E in the usual manner, patching where necessary using $P_\pm$.

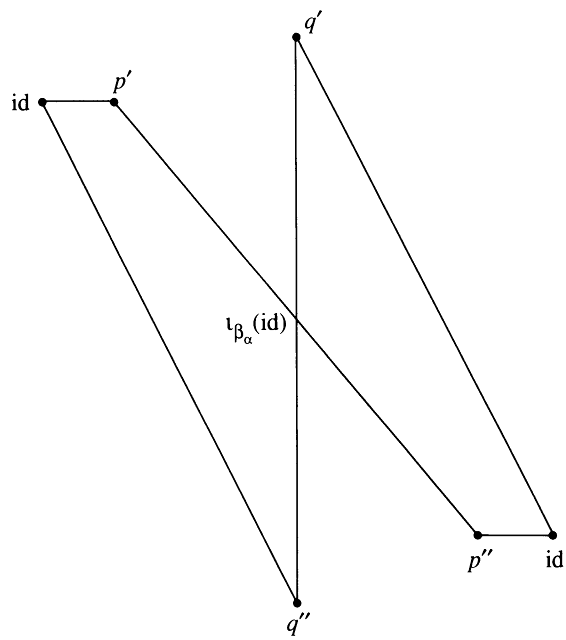


FIGURE 3.2.2

Let $J(p', q') = \frac{1}{4}(c_1(\det^1(\nu_{|_E}), \Phi) - \int_E \omega)$. We claim

$$J(p', q') = \tfrac{1}{2}(I(p) + I(q)),$$

where p' and q' are lifts of p and q respectively to $\widetilde{S}$. To see this, note that we could define $J(\iota_{\beta_\alpha} q', \iota_{\beta_\alpha} p')$ similarly, with our first arc from id to $\iota_{\beta_\alpha} q'$ in $\widetilde{T}_1^b$ and so on. The sum $J(p', q') + J(\iota_{\beta_\alpha} q', \iota_{\beta_\alpha} p')$ is independent of the choice of surface with boundary $\gamma'_{p', q'} \cup \gamma''_{p', q'} \cup \gamma'_{\iota_{\beta_\alpha} q', \iota_{\beta_\alpha} p'} \cup \gamma''_{\iota_{\beta_\alpha} q', \iota_{\beta_\alpha} p'}$ by the usual argument; in particular we may take two surfaces, appropriate for calculating $I(p)$ and $I(\iota_{\beta_\alpha} q)$ respectively. Clearly $J(p', q') = J(\iota_{\beta_\alpha} q', \iota_{\beta_\alpha} p')$, so the claim holds.

We may define $J(p', q')$ similarly for $p' \in \widetilde{T}_1^x \cap \iota_{\beta_\alpha} \widetilde{T}_2$ and $q' \in \iota_{\beta_\alpha} \widetilde{T}_1^y \cap \widetilde{T}_2$, where $x = -b$ and $y = -a$. Again, we obtain $J(p', q') = \frac{1}{2}(I(p) + +I(q))$.

Note that for each $p' \in \widetilde{T}_1^{a+b} \cap \iota_{\beta_\alpha} \widetilde{T}_2$, a surface E' suitable for calculating $I(p)$ tends to a surface E suitable for calculating either $I(q)$ for some q in T_1^a or T_1^{-b} (if $\beta_\alpha(b) = 1$) or $J(r', s')$ for some r', s' as above (if $\beta_\alpha(b) = -1$). We wish to compare the corresponding c_1-terms. Note that if $\beta_\alpha(b) = 1$, then the corresponding c_1-terms satisfy the desired relation trivially. Assume $\beta_\alpha(b) = -1$.

Let E be a surface suitable for calculating $J(r', s')$ for some r' and s' as above. We wish to compute $c_1(\det^1(\nu)_{|_E}, \Phi_{\lim}) - c_1(\det^1(\nu)_{|_E}, \Phi_J)$ where $\Phi_{\lim}$ is the trivialization which is the limit of the trivialization for computing $I(p)$ for the corresponding $p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2$ and Φ_J is the trivialization of described above for computing $J(r', s')$, where r' and s' are the appropriate points in ∂E. Since the surface involved is the same for each term, this difference is just the difference of the trivializations $c_1(\det^1(\nu)_{|_E}, \Phi_{\lim}) - c_1(\det^1(\nu)_{|_E}, \Phi_J) = \Phi_{\lim} - \Phi_J$.

Now the trivializations themselves agree by definition except for the patching process at r', ι_{β_α} id, s', id, $\tau(r')$, $\tau(\iota_{\beta_\alpha}\text{id}) = \iota_{\beta_\alpha}$ id, $\tau(s')$, and $\tau(\text{id}) = \text{id}$.

Hence the difference is just the sum of the homotopy classes of

$$\frac{1}{2}\left\{\lim_{t\to 1} h_t(P_+(\det{}^1(\eta^{a+b}_{1,r'}), \det{}^1(\iota_{\beta_\alpha}\eta_{2,r'}))) + \lim_{t\to 1} h_t(P_-(\det{}^1(\eta^{a+b}_{1,r'}), \det{}^1(\iota_{\beta_\alpha}\eta_{2,r'})))\right\}$$
$$+ \tfrac{1}{2}\{P_+(\det{}^1(\iota_{\beta_\alpha}\eta_{2,r'}), \det{}^1(\eta^a_{1,r'})) + P_-(\det{}^1(\iota_{\beta_\alpha}\eta_{2,r'}), \det{}^1(\iota_{\beta_\alpha}\eta^a_{1,r'}))\}$$

and so on, where all homotopy classes are in $\pi_1(S^1)$. But at $\tau(r')$, the corresponding homotopy class is that of

$$\frac{1}{2}\left\{\lim_{t\to 1} h_t(P_+(\det{}^1(\eta^{a+b}_{1,\tau(r')}), \det{}^1(\iota_{\beta_\alpha}\eta_{2,\tau(r')})). + \lim_{t\to 1} h_t(P_-(\det{}^1(\eta^{a+b}_{1,\tau(r')}), \det{}^1(\iota_{\beta_\alpha}\eta_{2,\tau(r')})))\right\}$$
$$+ \tfrac{1}{2}\{P_+(\det{}^1(\iota_{\beta_\alpha}\eta_{2,\tau(r')}), \det{}^1(\eta^a_{1,\tau(r')})) + P_-(\det{}^1(\iota_{\beta_\alpha}\eta_{2,\tau(r')}), \det{}^1(\iota_{\beta_\alpha}\eta^a_{1,\tau(r')}))\},$$

which is the negative of the homotopy class at r'. Similarly, the homotopy classes at ι_{β_α} id and $\tau(\iota_{\beta_\alpha}\text{ id}) = \iota_{\beta_\alpha}$ id cancel, as do those at s' and $\tau(s')$ and id and $\tau(\text{id}) = \text{id}$. Hence the difference of the trivializations is 0.

(1) = (2) now follows if we are careful to match up (r')'s and (s')'s properly according to the action of β_α on a, b and $a+b$.

Thus, we have

$$\langle Q_1^{a+b}, \iota_{\beta_\alpha}Q_2\rangle = \begin{cases} \langle Q_1^a, \iota_{\beta_\alpha}Q_2\rangle + \langle Q_1^{-b}, \iota_{\beta_\alpha}Q_2\rangle \\ \quad - \sum\limits_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2} \int_{E_p}\omega + \sum\limits_{p\in T_1^{a}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega \\ \quad + \sum\limits_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega \quad \text{if } \beta_\alpha(b) = 1, \\ \frac{1}{2}(\langle Q_1^a, \iota_{\beta_\alpha}Q_2\rangle + \langle Q_1^{-a}, \iota_{\beta_\alpha}Q_2\rangle \\ \quad + \langle Q_1^b, \iota_{\beta_\alpha}Q_2\rangle + \langle Q_1^{-b}, \iota_{\beta_\alpha}Q_2\rangle) \\ \quad - \sum\limits_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2} \int_{E_p}\omega + \dfrac{1}{2}\sum\limits_{p\in T_1^{a}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega \\ \quad + \dfrac{1}{2}\sum\limits_{p\in T_1^{-a}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega + \dfrac{1}{2}\sum\limits_{p\in T_1^{b}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega \\ \quad + \dfrac{1}{2}\sum\limits_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2} \int_{h_1(E_p)}\omega \quad \text{if } \beta_\alpha(b) = -1, \end{cases}$$

where the surfaces $h_1(E_p)$ are the images under h_1 of the surfaces used to calculate $\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}\int_{E_p}\omega$.

Before dealing with the integral terms, we rewrite $\langle Q_1^{-b}, \iota_{\beta_\alpha}Q_2\rangle$ and $\langle Q_1^{-a}, \iota_{\beta_\alpha}Q_2\rangle$ in terms of $\langle Q_1^b, \iota_{\beta_\alpha}Q_2\rangle$ and $\langle Q_1^a, \iota_{\beta_\alpha}Q_2\rangle$.

Since these numbers are independent of the choice of special isotopies used to put the various representation spaces into general position, we may assume that our isotopies were equivariant with respect to the $H^1(N_b;\mathbb{Z})$- and $H^1(N_a;\mathbb{Z})$-group actions, respectively. Thus, we may assume that $\sum_{p\in Q_1^{-b,-}\cap\iota_{\beta_\alpha}Q_2}\operatorname{sign}(p) = \sum_{p\in Q_1^{b,-}\cap\iota_{\beta_\alpha}Q_2}\operatorname{sign}(p)$ and $\sum_{p\in Q_1^{-a,-}\cap\iota_{\beta_\alpha}Q_2}\operatorname{sign}(p) = \sum_{p\in Q_1^{a,-}\cap\iota_{\beta_\alpha}T_2}\operatorname{sign}(p)$, since the points cover the same points in $R_{\mathrm{SO}(3)}$ and the covering map is orientation preserving.

We begin with $\langle Q_1^{-b}, \iota_{\beta_\alpha}Q_2\rangle$. Let E be a surface suitable for computing $I(p)$, where $p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2$. Then E is also suitable for computing $2J(x',p')$, where $x'\in\widetilde{T}_1^a\cap\iota_{\beta_\alpha}T_2\cap Z$ and p' is a lift of p to $\widetilde{S}$. Since $I(x)=0$, $2J(x',p')=I(p)$. Here p is viewed as an element of $\iota_yT_1^b\cap\iota_{\beta_\alpha}T_2$, where $y\in H^1(\Sigma;\mathbb{Z}/2)$ is chosen so that $\iota_y\,\mathrm{id}=x$. We wish to compare the c_1-terms for each viewpoint.

Since the surfaces are identical, the difference of the c_1-terms is just the difference of the trivializations $\Phi_{\mathrm{lim}}-\Phi_{P_\pm}$, where Φ_{lim} is the trivialization when p is viewed as a point in $T_1^{-b}\cap\iota_{\beta_\alpha}T_2$ and $\Phi_{P_\pm}$ is the trivialization for computing $2J(x',p')$. The trivializations agree except for the patching process at x'. Note that the difference in the c_1-terms is actually independent of the choice of surface E, since both values of $I(p)$ are and the integrals agree. Moreover, the patching process at x' used for the computation of $2J(x',p')$ is independent of the choice of E. It follows that the patching process at x' in Φ_{lim} is also independent of the choice of surface. Then it is clear that the patching process for Φ_{lim} is in fact independent of the point p'. It follows that the difference of the c_1-terms is in fact independent of p'. It is the average of the homotopy classes given by the path from $\eta^a_{1,x'}$ to $\iota_y\eta^b_{1,x'}$ in Φ_{lim} (for *any* $p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2$) followed by $P_\pm(\iota_y\eta^b_{1,x'},\eta^a_{1,x'})$.

Note that this is independent of β_α. It follows from [W, 3.48 and the proof of 4.10] that $(\langle Q_1^{-b},Q_2\rangle-\langle Q_1^b,Q_2\rangle)+\frac{1}{2}|H_1(N_b;\mathbb{Z})|=0$. This implies that $\Phi_{\mathrm{lim}}-\Phi_J=-1$ for all $p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2$.

A similar argument shows that the difference in the c_1-terms for points $p\in T_1^{-a}\cap\iota_{\beta_\alpha}T_2$ is $+1$. We obtain

$$\langle Q_1^{a+b},\iota_{\beta_\alpha}Q_2\rangle=\begin{cases}\langle Q_1^a,\iota_{\beta_\alpha}Q_2\rangle+\langle Q_1^b,\iota_{\beta_\alpha}Q_2\rangle\\ \quad-\displaystyle\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}\int_{E_p}\omega+\sum_{p\in T_1^a\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega\\ \quad+\displaystyle\sum_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega-\frac{1}{2}|H_1(N_b;\mathbb{Z})| & \text{if } \beta_\alpha(b)=1\\ \langle Q_1^a,\iota_{\beta_\alpha}Q_2\rangle+\langle Q_1^b,\iota_{\beta_\alpha}Q_2\rangle\\ \quad-\displaystyle\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}\int_{E_p}\omega+\frac{1}{2}\sum_{p\in T_1^a\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega\\ \quad+\displaystyle\frac{1}{2}\sum_{p\in T_1^{-a}\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega+\frac{1}{2}\sum_{p\in T_1^b\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega\\ \quad+\displaystyle\frac{1}{2}\sum_{p\in T_1^{-b}\cap\iota_{\beta_\alpha}T_2}\int_{h_1(E_p)}\omega+\frac{1}{4}|H_1(N_a;\mathbb{Z})|\\ \quad-\displaystyle\frac{1}{4}|H_1(N_b;\mathbb{Z})| & \text{if } \beta_\alpha(b)=-1.\end{cases}$$

We now turn out attention to the integrals. Let E be a surface suitable for computing $I(\rho)$ for some $p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2$. We wish to compare $\int_E \omega$ and $\int_{E_1} \omega$, where E_1 is the image of E under $\lim_{t\to 1} h_t$. We may assume E and E_1 differ by the tracks of the arcs γ_1 and $\iota_{\beta_\alpha}\gamma_1$ under $\{h_t\}$. These tracks lie in the sets $\{[\rho] \in R | \rho(\delta_i) = 1,\ 1 \le i \le g-1\}$ and $\{[\rho] \in R | (\iota_{\beta_\alpha}\rho)(\delta_i) = 1,\ 1 \le i \le g-1\}$, respectively. (Recall that δ_i bounds a disk transverse to the ith 1-handle in H_1.) Denote the corresponding surfaces by F_1' and F_2', so

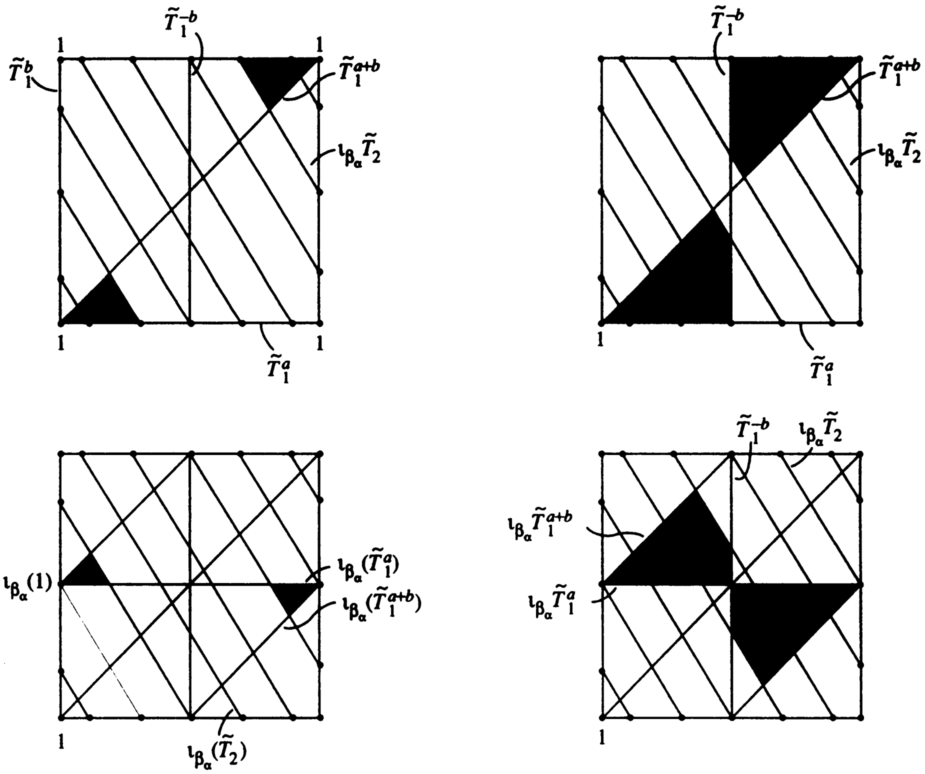
$$E_1 = E \cup F_1' \cup F_2' \quad \text{if } \beta_\alpha(b) = 1,$$
$$E_1 \cup F_2' = E \cup F_1' \quad \text{if } \beta_\alpha(b) = -1.$$

By naturality, we have $\int_{F_i'} \omega = \int_{F_i} \omega$. Here F_i is the projection of F_i' onto the set $\{[\rho] \in R | \rho_j(\delta_j) = \rho_j(\kappa_j) = 1,\ 1 \le j \le g-1\}$ where $\{\gamma_a, \gamma_b, \delta_1, \kappa_1, \dots, \delta_{g-1}, \kappa_{g-1}\}$ are a symplectic set of curves on Σ.

If $\beta_\alpha(a) = \beta_\alpha(b) = \beta_\alpha(a+b)$, Walker's proof shows that

$$\sum_{p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} \int_{F_1} \omega = \sum_{p \in T_1^{a+b} \cap \iota_{\beta_\alpha} T_2} \frac{1}{4} \int_{F_2} \omega$$
$$= \tfrac{1}{12}(|H_1(N_a; \mathbb{Z})| - |H_1(N_b; \mathbb{Z})|) + \tfrac{1}{2}|H_1(N_b; \mathbb{Z})|.$$

Otherwise:



Projections onto $\{[\rho] \in R | \rho(\delta_i) = \rho(\kappa_i) = 1\}$

FIGURE 3.2.3

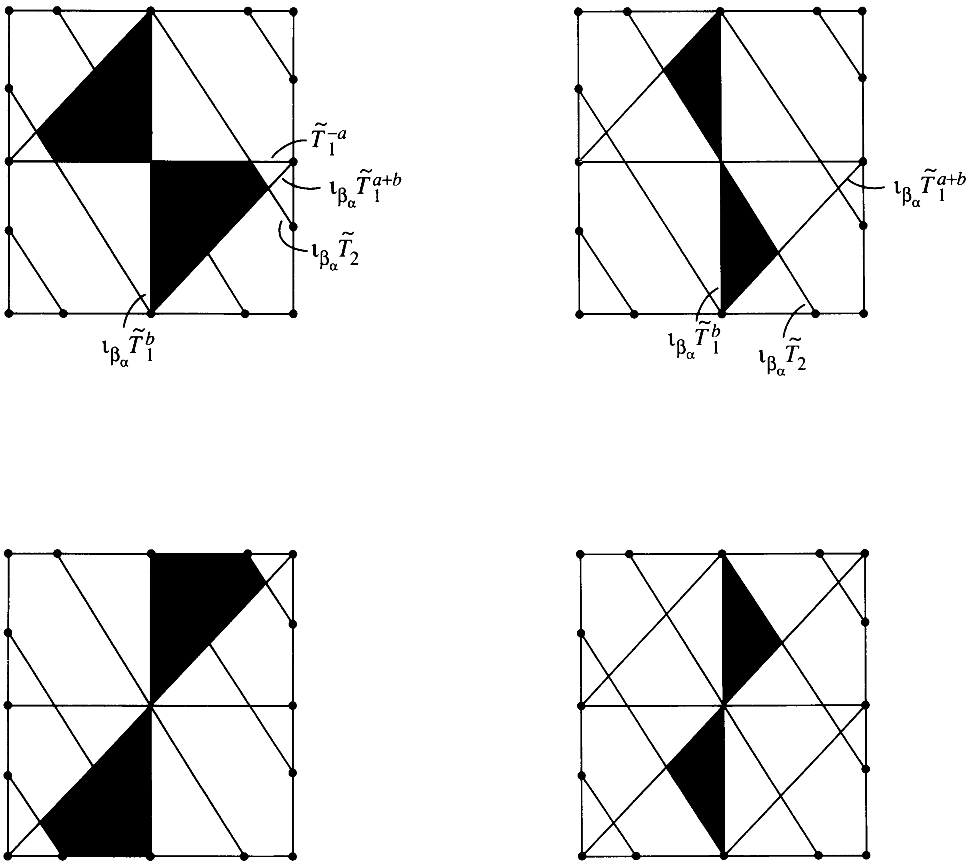


FIGURE 3.2.4

The surfaces F_1 are of the form shown in the upper pictures in Figure 3.2.3. If $\beta_\alpha(b) = 1$, the surfaces F_2 are of the form shown in the lower pictures. In either case, we have

$$\sum_{F_i} \frac{1}{4} \int_{F_i} \omega = \frac{n}{2} \left(\sum_{l=0}^{(v-1)/2} \frac{4}{v(u+v)} \left(\frac{2l+1}{2} \right)^2 + \sum_{l=(v+1)/2}^{(u+v-2)/2} \left(1 - \frac{4}{u(u+v)} \left(\frac{u+v}{2} - \frac{2l+1}{2} \right)^2 \right) \right)$$

where $|H_1(N_a\,;\mathbb{Z})| = nv$, $|H_1(N_b\,;\mathbb{Z})| = nv$, $(u\,,v) = 1$, and u and v are both odd. Similar formulas hold if v or u is even. If $\beta_\alpha(b) = -1$, the surfaces are of one of these forms. We have

$$\sum_{F_2} \frac{1}{4} \int_{F_2} \omega = \sum_{F_1} \frac{1}{4} \left(1 - \int_{F_1} \omega \right).$$

In Figures 3.2.3 and 3.2.4, examples of the various possibilities for the projections of the surfaces F_1 and F_2 are shown, and formulas for calculating the sums of the areas of the appropriate triangles are given. We see that

$$\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}\frac{1}{4}\int_{F_1}\omega=\frac{1}{12}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|)+\frac{1}{4}|H_1(N_b;\mathbb{Z})|,$$

while

$$\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}\frac{1}{4}\int_{F_2}\omega=\begin{cases}\frac{1}{12}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|)+\frac{1}{4}|H_1(N_b;\mathbb{Z})| & \text{if } \beta_\alpha(b)=1,\\ -\frac{1}{12}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|) \\ \quad -\frac{1}{4}|H_1(N_b;\mathbb{Z})|-\frac{1}{4}|H_1(N_{a+b};\mathbb{Z})| & \text{if } \beta_\alpha(b)=-1.\end{cases}$$

It follows that

$$\sum_{p\in Q_1^{a+b,-}\cap\iota_{\beta_\alpha}Q_2^-}\operatorname{sign}(p)+\sum_{p\in T_1^{a+b}\cap\iota_{\beta_\alpha}T_2}I(p)$$

$$=\begin{cases}\displaystyle\sum_{p\in Q_1^{a,-}\cap\iota_{\beta_\alpha}Q_2^-}\operatorname{sign}(p)+\sum_{p\in T_1^{a}\cap\iota_{\beta_\alpha}T_2}I(p)+\sum_{p\in Q_1^{b,-}\cap\iota_{\beta_\alpha}Q_2^-}\operatorname{sign}(p)\\ \displaystyle\quad+\sum_{p\in T_1^{b}\cap\iota_{\beta_\alpha}T_2}I(p)+\frac{1}{6}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|)\\ \quad-\frac{1}{2}|H_1(N_b;\mathbb{Z})|+\frac{1}{2}|H_1(N_b;\mathbb{Z})| \quad \text{if } \beta_a(b)=1\\ \displaystyle\sum_{p\in Q_1^{a,-}\cap\iota_{\beta_\alpha}Q_2^-}\operatorname{sign}(p)+\sum_{p\in T_1^{a}\cap\iota_{\beta_\alpha}T_2}I(p)+\sum_{p\in Q_1^{b,-}\cap\iota_{\beta_\alpha}Q_2^-}\operatorname{sign}(p)\\ \displaystyle\quad+\sum_{p\in T_1^{b}\cap\iota_{\beta_\alpha}T_2}I(p)+\frac{1}{4}|H_1(N_a;\mathbb{Z})|-\frac{1}{4}|H_1(N_b;\mathbb{Z})|\\ \quad+\frac{1}{6}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|)+\frac{1}{2}|H_1(N_b;\mathbb{Z})|\\ \quad-\frac{1}{4}|H_1(N_{a+b};\mathbb{Z})| \quad \text{if } \beta_\alpha(b)=-1.\end{cases}$$

Finally, we obtain

$$\overline{\lambda}_\alpha(N_{a+b})=\overline{\lambda}_\alpha(N_a)+\overline{\lambda}_\alpha(N_b)+\tfrac{1}{6}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|)$$

for all $\alpha\in H$. This proves the lemma.

Proof of Theorem. Any $\mathbb{Q}HS\,M$ is of the form N_a, where K is a knot in M and a is the appropriate class in the first homology group of a tubular neighborhood of K. Choosing K appropriately, we may assume that a has even order in $H_1(M;\mathbb{Z})$.

Note that $\langle a, b\rangle=1$ implies that exactly one of $H^2(N_a;\mathbb{Z}/2)$, $H^2(N_b;\mathbb{Z}/2)$, and $H^2(N_{a+b};\mathbb{Z}/2)$ is isomorphic to H. Since a has even order in $H_1(M;\mathbb{Z})$, we must have $H\cong H^2(N_a;\mathbb{Z}/2)$.

If $\alpha=(\alpha', 1)$ in H, where $\alpha'\in H^2(N_b;\mathbb{Z}/2)\cong H^2(N_{a+b};\mathbb{Z}/2)$, this means that we have shown

$$\overline{\lambda}_{\alpha'}(N_{a+b})=\overline{\lambda}_\alpha(N_a)+\overline{\lambda}_{\alpha'}(N_b)+\tfrac{1}{6}(|H_1(N_a;\mathbb{Z})|-|H_1(N_b;\mathbb{Z})|).$$

But applying the lemma to $(\alpha', 0)$, we obtain

$$\overline{\lambda}_{\alpha'}(N_{a+b}) = \overline{\lambda}_{(\alpha',0)}(N_a) + \overline{\lambda}_{\alpha'}(N_b) + \tfrac{1}{6}(|H_1(N_a;\mathbb{Z})| - |H_1(N_b;\mathbb{Z})|).$$

It follows that $\overline{\lambda}_\alpha(N_a) = \overline{\lambda}_{(\alpha',0)}(N_a)$.

Now choose another knot K' in M such that $M = N_{a'}$, where a' has even order in $H_1(M;\mathbb{Z})$. Inductively, we obtain $\overline{\lambda}_\alpha(N_a) = \overline{\lambda}_0(N_a)$ for all $\alpha \in H^2(N_a;\mathbb{Z}/2)$ and hence $\lambda_\alpha(N_a) = \lambda_0(N_a)$. Since $\lambda_0(N_a) = \frac{1}{|H^1(M;\mathbb{Z}/2)|}\lambda_{SU(2)}(N_a)$ by Corollary 3.2, this proves the theorem. □

4. Invariants for some other low-rank groups

In this chapter we define invariants of M counting $SU(2)\times S^1$-, U(2)-, Spin(4)-, and SO(4)-representations. We begin with $SU(2)\times S^1$.

4.1. The $SU(2)\times S^1$-invariant. Let M be a $\mathbb{Q}HS$. We define an invariant $\lambda_{SU(2)\times S^1}(M)$ of M and show $\lambda_{SU(2)\times S^1}(M) = |H_1(M;\mathbb{Z})|\lambda_{SU(2)}(M)$. We define numbers $I(p)$ for $p \in T_{1,SU(2)\times S^1} \cap T_{2,SU(2)\times S^1}$ such that

(5)
$$\lambda_{SU(2)\times S^1}(M) = \frac{\sum_{p\in Q^-_{1,SU(2)\times S^1}\cap Q^-_{2,SU(2)\times S^1}} \operatorname{sign}(p) + \sum_{p\in T_{1,SU(2)\times S^1}\cap T_{2,SU(2)\times S^1}} I(p)}{|H_1(M;\mathbb{Z})|}$$

is a well-defined invariant of M. We will need

Lemma 4.1. $\det^1(\nu_{SU(2)\times S^1})$ *is the induced bundle* $\pi^*\det^1(\nu_{SU(2)})$, *where* $\pi\colon \widetilde{S}_{SU(2)\times S^1} \to \widetilde{S}_{SU(2)}$ *is the projection. Moreover,*

$$\det{}^1(\eta_{j,SU(2)}\times S^1) = \pi^*\det{}^1(\eta_{j,SU(2)}),$$

and $\omega_{B'} = \pi^*\omega_B$, *where* B *is the* Ad-*invariant, positive definite bilinear form on* $\mathfrak{su}(2)$ *and* $B' = B\oplus B_{S^1}$ *is that on* $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$.

This is proven as Lemma 2.5. We obtain

Corollary 4.2. *The first Chern class* $c_1(\det^1(\nu_{SU(2)\times S^1}))$ *of* $\det^1(\nu_{SU(2)\times S^1})$ *is represented by a multiple* $\omega_{SU(2)\times S^1}$ *of* $\omega_{B'}$. *In fact,* $\omega_{SU(2)\times S^1} = \pi^*\omega_{SU(2)}$.

It follows that we may define

$$I(p) = \begin{cases} \frac{1}{2}\left(c_1(\det^1(\nu_{SU(2)\times S^1})|_E, \Phi) - \int_E \omega_{SU(2)\times S^1}\right) & \text{if } p\notin Z_{SU(2)\times S^1}, \\ 0 & \text{if } p \in Z_{SU(2)\times S^1} \end{cases}$$

for $p\in T_{1,SU(2)\times S^1}\cap T_{2,SU(2)\times S^1}$, where E and Φ are chosen as in the SO(3) case, replacing $\widetilde{S}_{SO(3)}$ and $\widetilde{T}_{j,SO(3)}$ with $\widetilde{S}_{SU(2)\times S^1}$ and $\widetilde{T}_{j,SU(2)\times S^1}$. These numbers are well defined by the usual argument. Moreover, the arguments of §2 and [W, II.B] show that $\lambda_{SU(2)\times S^1}$ is a well-defined invariant of M. We show

Theorem 4.3. $\lambda_{SU(2)\times S^1}(M) = |H_1(M;\mathbb{Z})|\lambda_{SU(2)}(M)$.

Proof. Note that $R_{SU(2)\times S^1} = R_{SU(2)}\times R_{S^1}$, and $Q_{j,SU(2)\times S^1} = Q_{j,SU(2)}\times Q_{j,S^1}$. Moreover, $Q_{1,SU(2)}\times Q_{1,S^1}\cap Q_{2,SU(2)}\times Q_{2,S^1}$ is just the product $(Q_{1,SU(2)}\cap Q_{2,SU(2)})\times(Q_{1,S^1}\cap Q_{2,S^1})$, at least before isotopy.

Note that Q_{1,S^1} and Q_{2,S^1} are already in general position in R_{S^1}, since

$H_1(M;\mathbb{Z})$ is finite. It follows that we may choose our special isotopy h: $R_{\mathrm{SU}(2)\times S^1}\times I\to R_{\mathrm{SU}(2)\times S^1}$ to be the identity on R_{S^1}. Assume that $Q_{1,\mathrm{SU}(2)\times S^1}$ has been isotoped into general position with respect to $Q_{2,\mathrm{SU}(2)\times S^1}$ by such an isotopy.

Let $\overline{p}\in Q^-_{1,\mathrm{SU}(2)\times S^1}\cap Q^-_{2,\mathrm{SU}(2)\times S^1}$, and let $p=\pi(\overline{p})$, where π is the projection $R_{\mathrm{SU}(2)\times S^1}\to R_{\mathrm{SU}(2)}$. We have

$$\begin{aligned}\operatorname{sign}(\overline{p}) &= \frac{[Q_{1,\mathrm{SU}(2)\times S^1}][Q_{2,\mathrm{SU}(2)\times S^1}]}{[R_{\mathrm{SU}(2)\times S^1}]}\\ &= \frac{[Q_{1,\mathrm{SU}(2)}][Q_{1,S^1}][Q_{2,\mathrm{SU}(2)}][Q_{2,S^1}]}{[R_{\mathrm{SU}(2)}][R_{S^1}]}\\ &= (-1)^{g(3g-3)}\frac{[Q_{1,\mathrm{SU}(2)}][Q_{2,\mathrm{SU}(2)}][Q_{1,S^1}][Q_{2,S^1}]}{[R_{\mathrm{SU}(2)}][R_{S^1}]}\\ &= \operatorname{sign}(p)\frac{[Q_{1,S^1}][Q_{2,S^1}]}{[R_{S^1}]}.\end{aligned}$$

But

$$\frac{[Q_{1,S^1}][Q_{2,S^1}]}{[R_{S^1}]}=\frac{[T_{1,S^1}][T_{2,S^1}]}{[S_{S^1}]}=\frac{[\widetilde{T}_{1,S^1}][\widetilde{T}_{2,S^1}]}{[\widetilde{S}_{S^1}]},$$

which is 1 by definition. Hence $\operatorname{sign}(\overline{p})=\operatorname{sign}(p)$.

Now let $\overline{q}\in T_{1,\mathrm{SU}(2)\times S^1}\cap T_{2,\mathrm{SU}(2)\times S^1}$, and let $q=\pi(\overline{q})$. Fix a surface E for calculating $I(\overline{q})$. By Lemma 4.1, $\det^1(\nu_{\mathrm{SU}(2)\times S^1})_{|E}=\pi^*\det^1(\nu_{\mathrm{SU}(2)})_{|\pi(E)}$. Hence $c_1(\det^1(\nu_{\mathrm{SU}(2)\times S^1})_{|E})=c_1(\det^1(\nu_{\mathrm{SU}(2)})_{|E})$.

Also by Lemma 4.1, π takes $\det^1(\eta_{j,\mathrm{SU}(2)\times S^1})_{|E})$ to $\det^1(\eta_{j,\mathrm{SU}(2)})_{|E})$. Moreover, π maps fibers isomorphically, so the patching procedures at $1_{\mathrm{SU}(2)\times S^1}$, $\overline{q}'$ and $\overline{q}''$ agree with those at $1_{\mathrm{SU}(2)}$, q', and q''. Hence $c_1(\det^1(\nu_{\mathrm{SU}(2)\times S^1})_{|E}),\Phi)$ $=c_1(\det^1(\nu_{\mathrm{SU}(2)})_{|\pi(E)}),\Phi)$.

Finally, the degree of the map $E\to\pi(E)$ is clearly one, so we have

$$\int_E\omega_{\mathrm{SU}(2)\times S^1}=\int_{\pi(E)}\omega_{\mathrm{SU}(2)}.$$

It follows that $I(\overline{q})=I(q)$.

Now note that $Q_{1,S^1}\cap Q_{2,S^1}$ consists of $|H_1(M;\mathbb{Z})|$ points. Hence there are $|H_1(M;\mathbb{Z})|$ points in $Q_{1,\mathrm{SU}(2)\times S^1}\cap Q_{2,\mathrm{SU}(2)\times S^1}$ over each point in $Q_{1,\mathrm{SU}(2)}\cap Q_{2,\mathrm{SU}(2)}$. Thus,

$$\begin{aligned}&\sum_{\overline{p}\in Q_{1,\mathrm{SU}(2)\times S^1}\cap Q_{2,\mathrm{SU}(2)\times S^1}}\operatorname{sign}(\overline{p})+\sum_{\overline{p}\in T_{1,\mathrm{SU}(2)\times S^1}\cap T_{2,\mathrm{SU}(2)\times S^1}}I(\overline{p})\\ &\quad=|H_1(M;\mathbb{Z})|\left(\sum_{p\in Q_{1,\mathrm{SU}(2)}\cap Q_{2,\mathrm{SU}(2)}}\operatorname{sign}(p)+\sum_{p\in T_{1,\mathrm{SU}(2)}\cap T_{2,\mathrm{SU}(2)}}I(p)\right).\end{aligned}$$

4.2. **The** U(2)-**invariant.** We define an invariant $\lambda_{\mathrm{U}(2)}(M)$ of the $\mathbb{Q}HS\,M$ and show $\lambda_{\mathrm{U}(2)}(M)=|H_1(M;\mathbb{Z})|\lambda_{\mathrm{SU}(2)}(M)$.

Recall that $\mathrm{SU}(2)\times S^1\subset\mathrm{Spin}(4)$ is the double cover of $\mathrm{U}(2)\subset\mathrm{SO}(4)$. Exactly as in the SO(3) case, we obtain

Lemma 4.4. $\det^1(\nu_{\mathrm{SU}(2)\times S^1})$ *is the induced bundle* $2^*\det^1(\nu_{\mathrm{U}(2)})$ *where* $2\colon \widetilde{S}_{\mathrm{SU}(2)} \to \widetilde{S}_{\mathrm{SU}(2)}$ *is the map of degree* 2 *which is the restriction to* $S^1_0 \times S^1$ *of the map* $R_{\mathrm{SU}(2)\times S^1} \to R_{\mathrm{U}(2)}$. *Moreover* $\det^1(\eta_{j,\mathrm{SU}(2)\times S^1}) = 2^*\det^1(\eta_{j,\mathrm{U}(2)})$.

Corollary 4.5. *The first Chern class* $c_1(\det^1(\nu_{\mathrm{SU}(2)}))$ *of* $\det^1(\nu_{\mathrm{SU}(2)})$ *is represented by a multiple* $\omega_{\mathrm{U}(2)}$ *of* $\omega_{B'}$.

Thus, we may again define

$$I(p) = \begin{cases} \frac{1}{2}\left(c_1(\det^1(\nu_{\mathrm{U}(2)})_{|E}, \Phi) - \int_E \omega_{\mathrm{U}(2)}\right) & \text{if } p \notin Z_{\mathrm{U}(2)}, \\ 0 & \text{if } p \in Z_{\mathrm{U}(2)} \end{cases}$$

for $p \in T_{1,\mathrm{U}(2)} \cap T_{2,\mathrm{U}(2)}$, where E and Φ are chosen as in the $\mathrm{SU}(2)$ case. Again, we obtain a well-defined invariant

$$(6) \qquad \lambda_{\mathrm{U}(2)}(M) = \frac{\sum_{p\in Q^-_{1,\mathrm{U}(2)}\cap Q^-_{2,\mathrm{U}(2)}} \operatorname{sign}(p) + \sum_{p\in T_{1,\mathrm{U}(2)}\cap T_{2,\mathrm{U}(2)}} I(p)}{|H_1(M;\mathbb{Z})|}.$$

We show

Theorem 4.6. $\lambda_{\mathrm{U}(2)}(M) = |H_1(M;\mathbb{Z})|\lambda_{\mathrm{SU}(2)}(M)$.

Proof. Note that there exists a map $d\colon \mathrm{U}(2) \to \mathrm{SO}(3)$ such that the diagram

$$\begin{array}{ccc} \mathrm{SU}(2)\times S^1 & \xrightarrow{\pi} & \mathrm{SU}(2) \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2} \\ \mathrm{U}(2) & \xrightarrow{d} & \mathrm{SO}(3) \end{array}$$

commutes. Here, the vertical maps are induced by the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. To see this, recall that $\mathrm{U}(2)$ is just $\mathrm{SU}(2)\times S^1/(x,y)\sim(-x,-y)$. We send $[(x,y)]$ to $[x]$. This induces a diagram

$$\begin{array}{ccc} R_{\mathrm{SU}(2)\times S^1} & \xrightarrow{\pi} & R_{\mathrm{SU}(2)} \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2} \\ R_{\mathrm{U}(2)} & \xrightarrow{d} & R_{\mathrm{SO}(3)} \end{array}$$

We claim that a special isotopy h of $R_{\mathrm{SO}(3)}$ induces a special isotopy of $R_{\mathrm{U}(2)}$ such that $d\overline{h}(R_{\mathrm{U}(2)}) = h(R_{\mathrm{SO}(3)})$. Let h be a special isotopy of $R_{\mathrm{SO}(3)}$, and let h' be the induced special isotopy of $R_{\mathrm{SU}(2)}$. Then $\overline{h} = h'\times \mathrm{id}$ is a special isotopy of $R_{\mathrm{SU}(2)}\times R_{S^1} = R_{\mathrm{SU}(2)\times S^1}$. Moreover, $\overline{h}$ is equivariant with respect to the $H^1(M;\mathbb{Z}/2)$-action, since

$$\begin{aligned}(h'\times\mathrm{id})(\imath_{\beta_\alpha}(U\times V)) &= (h'\times\mathrm{id})(\imath_{\beta_\alpha}U\times \imath_{\beta_\alpha}V) \\ &= \imath_{\beta_\alpha}h'(U)\times \imath_{\beta_\alpha}V\end{aligned}$$

for an open set $U\times V$ in $R_{\mathrm{SU}(2)}\times R_{S^1}$ which maps homeomorphically onto its image in $R_{\mathrm{U}(2)}$. Hence $\overline{h}$ descends to a special isotopy of $R_{\mathrm{U}(2)}$.

Assume all representation spaces have been put into general position using special isotopies induced from that of $R_{\mathrm{SO}(3)}$. Note that $\operatorname{sign}(p) = \operatorname{sign}(d(p))$

for $p \in Q^-_{1,\mathrm{U}(2)} \cap Q^-_{2,\mathrm{U}(2)}$, since the vertical maps in the second diagram above are orientation preserving and

$$\operatorname{sign}(\overline{p}) = \operatorname{sign}(p) \quad \text{for } \overline{p} \in Q^-_{1,\mathrm{SU}(2)\times S^1} \cap \imath_{\beta_\alpha} Q^-_{2,\mathrm{SU}(2)\times S^1}$$

by Theorem 4.3.

We have a diagram

$$\begin{array}{ccc} \widetilde{S}_{\mathrm{SU}(2)\times S^1} & \xrightarrow{\pi} & \widetilde{S}_{\mathrm{SU}(2)} \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2} \\ \widetilde{S}_{\mathrm{U}(2)} & \xrightarrow{d} & \widetilde{S}_{\mathrm{SO}(3)} \end{array}$$

Here, the map d is simply a projection, and it is clear that d is a bundle map. We obtain $\det^1(\nu_{\mathrm{U}(2)}) = d^* \det^1(\nu_{\mathrm{SO}(3)})$. Then an argument similar to that of the proof of Theorem 4.3 shows $I(p) = I(d(p))$ for all $p \in T_{1,\mathrm{U}(2)} \cap T_{2,\mathrm{U}(2)}$.

Finally, note that if all special isotopies are induced as described above, we have

$$\begin{array}{ccc} Q_{1,\mathrm{SU}(2)\times S^1} \cap Q_{2,\mathrm{SU}(2)\times S^1} & \xrightarrow{\pi} & Q_{1,\mathrm{SU}(2)} \cap Q_{2,\mathrm{SU}(2)} \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2} \\ Q_{1,\mathrm{U}(2)} \cap Q_{2,\mathrm{U}(2)} & \xrightarrow{d} & Q_{1,\mathrm{SO}(3)} \cap Q_{2,\mathrm{SO}(3)} \end{array}$$

Then counting inverse images at each stage, we see that each element of $Q_{1,U(2)} \cap Q_{2,U(2)}$ is covered by $|H_1(M;\mathbb{Z})|$ points. The theorem follows. □

4.3. **The** Spin(4)**-invariant.** We define an invariant $\lambda_{\mathrm{Spin}(4)}(M)$ of M and show $\lambda_{\mathrm{Spin}(4)}(M) = |H_1(M;\mathbb{Z})|(\lambda_{\mathrm{SU}(2)}(M))^2$.

Note that the stratification of $R_{\mathrm{Spin}(4)}$ is more complicated than in previous cases. Thus, we must be a bit more careful.

To start with, we must decide what isotopies are allowable. Note that $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$, so we may identify the representation spaces $R_{\mathrm{Spin}(4)} = R_{\mathrm{SU}(2)} \times R_{\mathrm{SU}(2)}$, and so on. Define an isotopy of $R_{\mathrm{Spin}(4)}$ to be *special* if it is the product of special isotopies of $R_{\mathrm{SU}(2)}$. Since $Q_{1,\mathrm{SU}(2)}$ and $Q_{2,\mathrm{SU}(2)}$ can be put into general position by a special isotopy of $R_{\mathrm{SU}(2)}$, it is clear that $Q_{1,\mathrm{SU}(2)} \times Q_{1,\mathrm{SU}(2)}$ can be put into general position with respect to $Q_{2,\mathrm{SU}(2)} \times Q_{2,\mathrm{SU}(2)}$ using a special isotopy of $R_{\mathrm{Spin}(4)}$.

We need to define correction terms for all of the various strata of $R_{\mathrm{Spin}(4)}$. Let $\pi_i\colon R_{\mathrm{SU}(2)} \times R_{\mathrm{SU}(2)} \to R_{\mathrm{SU}(2)}$ be the projection onto the ith factor. We define

$$I(p) = \begin{cases} (-1)^{g-1} I(\pi_1(p)) \operatorname{sign}(\pi_2(p)) & \text{if } p \in S_{\mathrm{SU}(2)} \times R_{\mathrm{SU}(2)}, \\ (-1)^{g-1} \operatorname{sign}(\pi_1(p)) I(\pi_2(p)) & \text{if } p \in R_{\mathrm{SU}(2)} \times S_{\mathrm{SU}(2)}, \\ (-1)^{g-1} I(\pi_1(p)) I(\pi_2(p)) & \text{if } p \in S_{\mathrm{SU}(2)} \times S_{\mathrm{SU}(2)}, \\ 0 & \text{if } p \in Z_{\mathrm{SU}(2)} \times R_{\mathrm{SU}(2)} \text{ or } p \in R_{\mathrm{SU}(2)} \times Z_{\mathrm{SU}(2)}. \end{cases}$$

Finally, set

$$\lambda_{\mathrm{Spin}(4)}(M) = (-1)^{g-1} \frac{\sum_{p \in Q^-_{1,\mathrm{Spin}(4)} \cap Q^-_{2,\mathrm{Spin}(4)}} \operatorname{sign}(p) + \sum_{p \in T_{1,\mathrm{Spin}(4)} \cap T_{2,\mathrm{Spin}(4)}} I(p)}{|H_1(M;\mathbb{Z})|}.$$

We show

Theorem 4.7. $\lambda_{\mathrm{Spin}(4)}(M) = |H_1(M;\mathbb{Z})|(\lambda_{\mathrm{SU}(2)}(M))^2$. *In particular,* $\lambda_{\mathrm{Spin}(4)}(M)$ *is a well-defined invariant of the* $\mathbb{Q}HS$ M.

Proof. Let $p \in Q^-_{1,\mathrm{Spin}(4)} \cap Q^-_{2,\mathrm{Spin}(4)}$. We wish to compare $\mathrm{sign}(p)$ with $\mathrm{sign}(\pi_1(p))\,\mathrm{sign}(\pi_2(p))$, where π_i is the projection $R_{\mathrm{Spin}(4)} \to R_{\mathrm{SU}(2)}$ onto the ith factor. We have

$$\begin{aligned}\mathrm{sign}(p) &= \frac{[Q_{1,\mathrm{Spin}(4)}][Q_{2,\mathrm{Spin}(4)}]}{[R_{\mathrm{Spin}(4)}]}\\ &= \frac{[Q_{1,\mathrm{SU}(2)}][Q'_{1,\mathrm{SU}(2)}][Q_{2,\mathrm{SU}(2)}][Q'_{2,\mathrm{SU}(2)}]}{[R_{\mathrm{SU}(2)}][R'_{\mathrm{SU}(2)}]}\\ &= (-1)^{(3g-3)(3g-3)}\frac{[Q_{1,\mathrm{SU}(2)}][Q_{2,\mathrm{SU}(2)}][Q'_{1,\mathrm{SU}(2)}][Q'_{2,\mathrm{SU}(2)}]}{[R_{\mathrm{SU}(2)}][R'_{\mathrm{SU}(2)}]}\\ &= (-1)^{g-1}\,\mathrm{sign}(\pi_1(p))\,\mathrm{sign}(\pi_2(p)).\end{aligned}$$

It follows that

$$\begin{aligned}&(-1)^{g-1}\sum_{p\in Q^-_{1,\mathrm{Spin}(4)}\cap Q^-_{2,\mathrm{Spin}(4)}}\mathrm{sign}(p) + \sum_{p\in T_{1,\mathrm{Spin}(4)}\cap T_{2,\mathrm{Spin}(4)}} I(p)\\ &= \left(\sum_{Q^-_{1,\mathrm{SU}(2)}\cap Q^-_{2,\mathrm{SU}(2)}}\mathrm{sign}(p) + \sum_{T_{1,\mathrm{SU}(2)}\cap T_{2,\mathrm{SU}(2)}} I(p)\right)\\ &\quad\cdot\left(\sum_{Q'^-_{1,\mathrm{SU}(2)}\cap Q'^-_{2,\mathrm{SU}(2)}}\mathrm{sign}(p) + \sum_{T'_{1,\mathrm{SU}(2)}\cap T'_{2,\mathrm{SU}(2)}} I(p)\right).\end{aligned}$$

This shows $\lambda_{\mathrm{Spin}(4)}(M) = |H_1(M;\mathbb{Z})|(\lambda_{\mathrm{SU}(2)}(M))^2$. Since this relationship is independent of the choice of special isotopy, the choice of orientations of $\widetilde{T}_1$ and $\widetilde{T}_2$, and the choice of Heegaard decomposition of M, we see that $\lambda_{\mathrm{Spin}(4)}(M)$ is a well-defined invariant of M. □

We remark that there should be the usual geometric descriptions of $(-1)^{g-1}\,\mathrm{sign}(\pi_1(p))I(\pi_2(p))$, $(-1)^{g-1}I(\pi_1(p))\,\mathrm{sign}(\pi_2(p))$ and $(-1)^{g-1}I(\pi_1(p)) \times I(\pi_2(p))$ for $p \in Q^-_{1,\mathrm{SU}(2)} \times T_{1,\mathrm{SU}(2)} \cap Q^-_{2,\mathrm{SU}(2)} \times T_{2,\mathrm{SU}(2)}$, $p \in T_{1,\mathrm{SU}(2)} \times Q^-_{1,\mathrm{SU}(2)} \cap T_{2,\mathrm{SU}(2)} \times Q^-_{2,\mathrm{SU}(2)}$, and $p \in T_{1,\mathrm{Spin}(4)} \cap T_{2,\mathrm{Spin}(4)}$ in terms of the determinant line bundles of the various Zariski normal bundles. For $p \in Q^-_{1,\mathrm{SU}(2)} \times T_{1,\mathrm{SU}(2)} \cap Q^-_{2,\mathrm{SU}(2)} \times T_{2,\mathrm{SU}(2)}$ and $p \in T_{1,\mathrm{SU}(2)} \times Q^-_{1,\mathrm{SU}(2)} \cap T_{2,\mathrm{SU}(2)} \times Q^-_{2,\mathrm{SU}(2)}$, arguments similar to those for $\mathrm{SU}(2) \times S^1$- and $\mathrm{U}(2)$-theory, where extra care is taken with signs, will show that the correction terms defined in this way agree with those coming from $\mathrm{SU}(2)$-theory as given above. For $p \in T_{1,\mathrm{Spin}(4)} \cap T_{2,\mathrm{Spin}(4)}$ this would involve much more work.

4.4. **The SO(4)-invariant.** Our final task is to define an invariant $\lambda_{\mathrm{SO}(4)}(M)$ and show

Theorem 4.8. $\lambda_{SO(4)}(M) = |H_1(M;\mathbb{Z})|(\lambda_{SU(2)}(M))^2$.

All proofs are completely analogous to other proofs in this paper. We therefore simply outline the approach.

As in §4.2, we have the diagram

$$\begin{array}{ccc} \mathrm{Spin}(4) & \xrightarrow{\pi_i} & SU(2)\times SU(2) \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2\times 2} \\ SO(4) & \xrightarrow{\pi_i} & SO(3)\times SO(3) \end{array}$$

and the induced diagrams of representation spaces. Let a *special* isotopy of $R_{SO(4)}$ be one which is induced from a special isotopy of $R_{SO(3)}$, as in §4.2.

Let $p \in Q_{1,S(O(2)\times O(2))} \cap Q_{2,S(O(2)\times O(2))}$. As in §2.2, we define

$$s(p) = \begin{cases} \frac{1}{2}\,\mathrm{sign}(\tilde{p}) & \text{if } p \in R^-_{S(O(2)\times O(2))}, \\ \frac{1}{4}\,\mathrm{sign}(\tilde{p}) & \text{if } p \in R_{\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2}, \end{cases}$$

where $\tilde{p}$ is a lift of p to a smooth cover $\widetilde{U}$ of a neighborhood U of p. Moreover, we define

$$I(p) = \begin{cases} (-1)^{g-1} I(\pi_1(p))\,\mathrm{sign}(\pi_2(p)) \\ \qquad\qquad \text{if } p \in \text{ the copy of } U(2) \text{ covering } S_{SO(3)} \times R_{SO(3)}, \\ (-1)^{g-1}\,\mathrm{sign}(\pi_1(p)) I(\pi_2(p)) \\ \qquad\qquad \text{if } p \in \text{ the copy of } U(2) \text{ covering } R_{SO(3)} \times S_{SO(3)}, \\ (-1)^{g-1} I(\pi_1(p)) I(\pi_2(p)) \quad \text{if } p \in S^-_{SO(4)} \text{ or } p \in S_{\mathbb{Z}/2\oplus\mathbb{Z}/2}, \\ 0 \quad \text{if } p \in R_{SU(2)}. \end{cases}$$

Finally, set

$$\lambda_{SO(4)} = (-1)^{g-1} \frac{\sum_{p\in Q^-_{1,SO(4)}\cap Q^-_{2,SO(4)}} \mathrm{sign}(p) + \sum_{p\in T_{1,SO(4)}\cap T_{2,SO(4)}} I(p)}{|H_1(M;\mathbb{Z})|}.$$

Arguments analogous to those in §§4.2 and 4.3 show that this defines an invariant of M and $\lambda_{SO(4)}(M) = |H_1(M;\mathbb{Z})|(\lambda_{SU(2)}(M))^2$.

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